

New approach to the microscopic theory of Fermi systems

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A new version of the microscopic theory of Fermi systems is proposed. A closed functional equation is obtained for the effective interaction potential R of the quasiparticles. Methods for calculating R in terms of the particle interaction potential are discussed. It is shown that in the well-known local approximation the functional equation for R reduces to an integro-differential equation. To within a few percent, this approximation reproduces the Monte Carlo equations of state for the model systems for which Monte Carlo results are known. The contribution of the Δ resonance degrees of freedom to the ground-state energy is analyzed. Problems of the microscopic calculation of the properties of phase transitions are discussed.

INTRODUCTION

This paper gives an exposition of a new version of microscopic theory for Fermi systems with arbitrary interaction between the particles. The task of any microscopic theory is to describe the properties of the considered system in terms of the parameters that determine its Hamiltonian. In many-body theory, several approaches associated with summation of infinite perturbation theories have so far been developed. One of them, the cluster expansion,¹ is designed to find the pair correlation function $g(r_{12})$, which determines the interaction energy of the system. The development of an algorithm for summing the diagrams of the cluster expansion for $g(r_{12})$ and methods of approximate solution of the resulting equations is the main content of FHNC (Fermi hypernetted chain) theory.² This approach has been widely and successfully applied to the description of the properties of various systems. But some diagrams, for example, in nuclear calculations beginning with the fifth order, as in any approach based on the use of perturbation theory, must be ignored, and this makes the approach vulnerable, especially when it is applied to the description of dense systems.

Another version of microscopic many-body theory—the coupled-cluster (exp S) theory—was proposed in Refs. 3–5. It operates with a wave function of the system. Exact solution of the problem in this approach involves the consideration of infinite systems of coupled equations that determine the contribution of the many-particle correlations. The standard method of solving this system is to truncate the hierarchy of cluster equations, the contribution of all clusters beginning with a certain number k being neglected. This leads to errors that are sometimes difficult to estimate.

The Monte Carlo approach⁶ is free of the shortcomings associated with the loss of the contribution of certain perturbation-theory graphs. In it, the variational problem of finding the minimum for the energy of the many-particle system, which consists of calculating multidimensional integrals, is solved by Monte Carlo methods. This method of calculation, which is very laborious, particularly for Fermi

systems, is regarded as the most accurate method for calculating the ground-state energy. Monte Carlo results for model Hamiltonians are treated as experimental data, and results obtained by other methods are usually compared with them. Unfortunately, the Monte Carlo method works best with simple potentials, and as yet there are no Monte Carlo results with realistic potentials. In addition, in this approach a small number of properties of the system are calculated, and there are great difficulties in calculating the spectrum of single-particle and collective excitations of the system and the parameters in the low-temperature expansion of the thermodynamic potential.

For 20 years, a different approach has been developed in the theory of an electron gas, in which the quasiparticle formalism developed by Hohenberg, Kohn, and Sham,^{7,8} is used to calculate the pair correlation function. The advantages of this approach are, first, its simplicity and transparency. Second, this approach makes it possible to use the achievements in Landau's theory of Fermi liquids^{9,10} and, if inhomogeneous systems are under consideration, the experience from the theory of finite Fermi systems.¹¹

The proposed new version of microscopic theory of Fermi systems is in the line of these approaches. The main tool of the new approach is the linear response function $\chi(1, 2)$ of the system and its functional derivatives $\delta\chi/\delta\rho$, $\delta^2\chi/\delta\rho\delta\rho$, etc. (ρ is the density of the system). The principal equation of the approach is a functional equation for the effective interaction potential R of the particles; this determines the difference between the system's response function χ and the response function χ_0 of noninteracting particles. Formally, this equation is equivalent to the system of equations of the cluster approach or the system of equations of the Green's-function method, but the closed form of the resulting equation and the simplicity of the interpretation makes it possible to develop relatively simple approximate methods of its solution. As a result, in the zeroth—local—approximation everything reduces to a single integro-differential equation, which can be solved by numerical methods. This zeroth approximation already reproduces the known Monte Carlo results for the ground-

state energy with an error of 2–5% in the entire range of variation of the density.^{12,13} Nevertheless, for some systems the accuracy of the zeroth approximation may be inadequate. Therefore a large proportion of the paper is devoted to the refinement of this approximation and the formulation of iterative computational procedures based on the use of the local approximation for the functional derivatives of χ . We emphasize that no perturbation-theory diagrams are ignored in either the zeroth or the following approximations. Moreover, at each stage of the calculations one of the basic consistency properties—the sum rule for the compressibility¹⁴—is satisfied.

One of the sections of the paper is devoted to microscopic calculation of the constants in Landau's Fermi-liquid theory in the framework of the proposed approach. Such calculations are not yet amenable to Monte Carlo methods. However, in our approach calculation of the effective mass M^* and of the constants f and g , which characterize the scattering amplitude of the quasiparticles at the Fermi surface, requires the solution of linear integral equations with known kernels expressed in terms of the effective potential R , which can be calculated separately, when the equation of state of the system is found. In the review, we also consider the problem of quantitative determination of the properties of the various phase transitions which occur in the system when its density is changed.

1. BASIC EQUATIONS

The method that we wish to describe is based on use of the response function. In this sense, it is a “descendant” of the approaches developed to calculate the energy of an electron gas in solid-state theory.^{8,14} However, in contrast to the remaining methods, this one permits achievement of the aim—a rigorous functional equation for the effective interaction potential R of the particles, which determines not only the response function but many other properties of the system. Moreover, the aim is achieved with extreme brevity—literally in a few steps.

We begin with the equation of state $\varepsilon(\rho)$ of a normal Fermi system whose particles interact by means of a scalar two-body potential \mathcal{V} . Then its energy E_0 can be expressed in terms of the density–density linear response function χ by the following formula¹⁴ (for details, see Appendix A):

$$E_0 = \tau + W,$$

where

$$W = W_H - \frac{1}{2} \int \int_C \mathcal{V}(\mathbf{r}_1 - \mathbf{r}_2) [\chi(\mathbf{r}_1, \mathbf{r}_2, \omega) + 2\pi\rho(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(i\omega)] d\mathbf{r}_1 d\mathbf{r}_2 \frac{d\omega}{2\pi i} \frac{dg_1}{g_1}. \quad (1)$$

The Hartree energy W_H is given by the well-known relation

$$W_H = \frac{1}{2} \int \int \rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\mathcal{V}(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad (2)$$

where τ is the energy of a system of noninteracting particles. The integral over the coupling constant g_1 is from zero to the actual value g , and the integral over ω is along the imaginary axis from $-i\infty$ to $+i\infty$.

We note immediately (for more details, see Sec. 4) that an expression like (1) is also obtained for nuclear one-boson exchange Hamiltonians. The only difference is that now the result also contains other response functions: spin–spin, χ_{ss} etc.

We recall that the density–density linear response function χ determines the change $\delta\rho$ in the density of the ground state of the system in a weak scalar external field δV_0 :

$$\delta\rho(\mathbf{r}_1, \omega) = \int \chi(\mathbf{r}_1, \mathbf{r}_2, \omega) \delta V_0(\mathbf{r}_2) d\mathbf{r}_2. \quad (3)$$

In fields with imaginary frequency $\omega = i\omega$, there is no energy transfer, and the ground state of the system remains the ground state. At the same time, the change in the internal energy of the system, $\delta^2 E_0 = (\tilde{0}|H|\tilde{0}) - (0|H|0)$, can be again expressed in terms of the response function χ :

$$\delta^2 E_0 = -\frac{1}{2} \int \delta V_0(\mathbf{r}, \omega) \chi(\mathbf{r}_1, \mathbf{r}_2, \omega) \delta V_0^*(\mathbf{r}_2, \omega) d\mathbf{r}_1 d\mathbf{r}_2 \quad (4)$$

(for more details, see Appendix A).

By means of (3), the external field δV_0 can be eliminated from the expression (4). For simplicity, we restrict ourselves for the time being to the homogeneous case. After transition to the Fourier components, the connection between $\delta\rho$ and δV_0 becomes algebraic:

$$\delta\rho(\mathbf{k}, \omega) = \chi(\mathbf{k}, \omega) \delta V_0(\mathbf{k}, \omega).$$

Then it follows from (3) and (4) that

$$\delta^2 E_0 = -(1/2) \delta\rho(\mathbf{k}, \omega) \chi^{-1}(\mathbf{k}, \omega) \delta\rho(-\mathbf{k}, -\omega)$$

or

$$\frac{\delta^2 E_0}{\delta\rho(\mathbf{k}, \omega) \delta\rho(-\mathbf{k}, -\omega)} = -\chi^{-1}(\mathbf{k}, \omega). \quad (5)$$

Note that (see Ref. 15)

$$\frac{\delta^2 \tau}{\delta\rho(\mathbf{k}, \omega) \delta\rho(-\mathbf{k}, -\omega)} = -\chi_0^{-1}(\mathbf{k}, \omega), \quad (6)$$

where $\chi_0(\mathbf{k}, \omega)$ is the response function of a system of noninteracting particles moving in the external self-consistent field $U = \delta W / \delta\rho$. This function can be readily calculated:¹⁴

$$\chi_0(\mathbf{k}, \omega) = \text{Sp} \int G_0(\mathbf{p} + \mathbf{k}/2, \varepsilon + \omega/2) \times G_0(\mathbf{p} - \mathbf{k}/2, \varepsilon - \omega/2) \frac{d^4 p}{(2\pi)^4 i}, \quad (7)$$

where $G_0(\mathbf{p}, \varepsilon) = (\varepsilon - U - p^2/2M)$ is the Green's function of a free particle, and M is its mass. If we introduce the notation $\omega = i\omega$, $z = \omega M/kp_F$, $y = k/2p_F$, then the integral (7) can be rewritten in the form

$$\chi_0(z, y) = -\frac{p_F M}{2\pi^2} \left[1 + \frac{1}{4} \left(\frac{1+z^2}{y} - y \right) \ln \frac{(1+y)^2 + z^2}{(1-y)^2 + z^2} - z \arctan(2z/(z^2 + y^2 - 1)) \right]. \quad (8)$$

Note that if in (1) we replace χ by the free function χ_0 , then we obtain the Hartree-Fock energy of the system. The difference between χ and χ_0 determines in (1) the correlation corrections to E_0 . The difference between χ and χ_0 can be characterized by a function R , which we shall call the effective particle interaction potential, introducing it by the formula

$$\chi^{-1}(\mathbf{k}, \omega) = \chi_0^{-1}(\mathbf{k}, \omega) - R(\mathbf{k}, \omega) \quad (9)$$

or, in more customary form,

$$\chi(\mathbf{k}, \omega) = \chi_0(\mathbf{k}, \omega) \kappa(\mathbf{k}, \omega), \quad (10)$$

where

$$\kappa(\mathbf{k}, \omega) = (1 - R(\mathbf{k}, \omega) \chi_0(\mathbf{k}, \omega))^{-1}. \quad (10')$$

In the random-phase approximation, R is replaced by the ordinary potential \mathcal{V} . To derive the basic equation of the entire method, the functional equation for the effective potential R , it remains to take one simple step—to substitute on the left-hand side of Eq. (1) the formula (1), and on the right-hand side the ansatz (9). Then after simple manipulations we arrive at a closed functional equation for the effective potential R :^{12,13,15}

$$R(\mathbf{k}, \omega) = \mathcal{V}(\mathbf{k}) - \frac{1}{2} \frac{\delta^2}{\delta \rho(\mathbf{k}, \omega) \delta \rho(-\mathbf{k}, -\omega)} \times \int \int_c \mathcal{V}(\mathbf{q}) \chi(\mathbf{q}, q_0) \frac{d^4 q dg}{(2\pi)^4 g}. \quad (11)$$

Note that the ansatz (9), in which all the influence of the interaction on the response function χ is attributed to the effective interaction potential R , is natural for the philosophy of the Hohenberg-Kohn-Sham approach^{7,8} adopted in solid-state theory. However, other ansatzes are also possible, for example, the Fermi-liquid ansatz,¹⁶ when χ_0 is taken to be the response function of a system of noninteracting quasiparticles with effective mass $M^* \neq M$, and then the interaction of the particles plays a dual role. On the one hand, it leads to a renormalization of the effective mass of the quasiparticles, while on the other it leads to the appearance of an effective but somewhat different interaction potential between the quasiparticles. This ansatz is more complicated than the ansatz (9), but it does have its advantages, which we shall not discuss here.

If we could solve the functional equation (11), then we would be able to calculate directly the compressibility of the system, $K = dP/d\rho$ (the pressure is $P = -\partial E_0/\partial V$), since, as is well known,¹⁴ it is related to the static limit:

$$\chi^{-1}(0, 0) = -K/\rho. \quad (12)$$

Substituting (9) here, we readily find

$$K = \frac{2}{3} \varepsilon_F^0 + \rho R(0, 0). \quad (13)$$

Calculating $R(0, 0)$, we must take the functional derivative $\delta/\delta \rho(0)$, and it is identical to $d/d\rho$, and therefore we obtain from (11) an integro-differential relation for the compressibility of the system [$q = (\mathbf{q}, q_0)$]:

$$K = \frac{2}{3} \varepsilon_F^0 - \frac{1}{2} \frac{d^2}{d\rho^2} \int \int_c \mathcal{V}(\mathbf{q}) \frac{\chi_0(q)}{(1 - R(q) \chi_0(q))} \frac{dg}{g} \frac{d^4 q}{(2\pi)^4}. \quad (13')$$

The functional equation (11) is a compact form of expression of a system of coupled equations for the effective potential R and its functional derivatives. This system, which is considered in more detail in Appendix B, is equivalent to the exact system of coupled equations for the many-particle Green's functions or the system of equations of coupled-cluster theory.^{3,4} But it is precisely the closed form of Eq. (11) that permits formulation of a new scheme of approximate solution of the fundamental system of equations of many-body theory. The scheme is valid not only for normal Fermi systems but also for superfluid Fermi systems and for Bose liquids.

2. APPROXIMATE SOLUTION OF THE FUNCTIONAL EQUATION FOR THE EFFECTIVE PARTICLE INTERACTION POTENTIAL

In the previous section, we obtained a closed functional equation for the effective particle interaction potential $R(k, \omega)$. This equation can be expanded into a hierarchy of equations that relate it to the functional derivatives $\delta R/\delta \rho$, $\delta^2 R/\delta \rho \delta \rho$, etc.

We begin with the first equation of this hierarchy (the details of the derivation are deferred to Appendix B):¹⁵

$$R(k) = \mathcal{V}(\mathbf{k}) - \frac{1}{2} \int \int_c \mathcal{V}(\mathbf{q}) [\kappa(\mathbf{q}) I^2(q, k) \kappa(q) + 2\kappa(q) I^1(q, k) T(q+k) I^1(k, q) \kappa(q) + 2\kappa(q) I^1(q, k) \kappa(q+k) O^1(k, q) \chi(q) + 2\chi(q) O^1(q, k) \kappa(q+k) I^1(k, q) \kappa(q) + 2\chi(q) O^1(q, k) \chi(q+k) O^1(k, q) \chi(q) + \chi(q) O^2(q, k) \chi(q)] \frac{d^4 q}{(2\pi)^4} \frac{dg}{g}. \quad (14)$$

Here, we have introduced the notation $T(q) = R(q)\kappa(q) = R(q)/(1 - R(q)\chi_0(q_1))$;

$$I^1(q, k) = \frac{\delta\chi_0(q)}{\delta\rho(k)}; \quad (15)$$

$$I^2(q, k) = \frac{\delta^2\chi_0(q)}{\delta\rho(k)\delta\rho(-k)};$$

$$O^1(q, k) = \frac{\delta R(q)}{\delta\rho(k)};$$

$$O^2(q, k) = \frac{\delta^2 R(q)}{\delta\rho(k)\delta\rho(-k)}.$$

We see that the potential R can be expressed in terms of the blocks $O^1 = \delta R/\delta\rho$ and $O^2 = \delta^2 R/\delta\rho\delta\rho$. For them we can also obtain equations if we again vary both sides of (14), but we then encounter new blocks O^3 , O^4 , etc. The upshot is an infinite hierarchy of equations relating different functional derivatives of the potential R . The most frequent device used to solve such systems consists of neglecting the contribution of all the many-particle blocks beginning with a certain order. As a result, the system is closed and can then be solved by numerical methods. We shall proceed differently.

We have already seen that the functional derivative with respect to ρ reduces to a partial derivative in the long-wavelength limit $k \rightarrow 0$. In the coordinate representation, this means that

$$\int \frac{\delta F(z)}{\delta\rho(x)} dx = \frac{dF(z)}{d\rho}. \quad (16)$$

For arbitrary k , there is no exact relationship between the variational derivative and the partial derivative. But one can find an approximate connection between them by noting that the basic properties of a system with strong interaction are local, i.e., an arbitrary functional $F(z)$ depends on the density ρ in the neighborhood of the same point z , and therefore in a homogeneous system we can write down the approximate relation

$$\frac{\delta F(z)}{\delta\rho(x)} = \delta(x - z) \frac{dF(z)}{d\rho}. \quad (17)$$

The relation (17), which is approximate, holds integrally, as can be seen from (16). We shall call (17) the local approximation, following the terminology of Ref. 17, where it was used for the first time. In general form, the procedure for solving the functional equation (11) is now as follows. The first k equations of the hierarchy are written down. The functional derivatives occurring in the last of them are calculated in accordance with the prescription of the local approximation (17), as a result of which these equations obviously close. The higher the number of the equation in which the approximation (17) is used, the more accurately the amplitude of R is determined, since the error is introduced by a correlation function of higher order. (We shall return to discussion of this question later.)

The calculations of Refs. 12 and 13 show that the use of the local approximation makes it possible to obtain very accurate results already in the first stage, when there remains only one equation. This is achieved as follows.

In accordance with (17), we write down an approximate equation for the second functional derivative of χ ($x = r, t$):

$$\frac{\delta^2\chi(x, x')}{\delta\rho(x_1)\delta\rho(x_2)} = \frac{1}{2} [\delta(x - x_1)\delta(x' - x_2) + \delta(x - x_2)\delta(x' - x_1)] \frac{d^2\chi(x, x')}{d\rho^2}. \quad (18)$$

Going over to the momentum representation, we find from this the approximate relation

$$\frac{\delta^2\chi(q)}{\delta\rho(k)\delta\rho(-k)} = \frac{d^2\chi(q + k)}{d\rho^2}, \quad (19)$$

which becomes exact in the limit $k \rightarrow 0$.

Substituting this relation (11), we arrive at a closed integro-differential equation for the effective particle interaction potential R . We denote this zeroth approximation to the exact effective potential R by R^0 . Then

$$R^0(k) = \mathcal{V}(k) - \frac{1}{2} \frac{d^2}{d\rho^2} \times \int \int_c \mathcal{V}(\mathbf{q}) \frac{\chi_0(q)}{(1 - R(q)\chi_0(q))} \frac{d^4q}{(2\pi)^4} \frac{dg}{g}. \quad (20)$$

As is seen from this, the effective potential R does not depend on the frequency k_0 , since the original interaction \mathcal{V} is unretarded. To improve the convergence of the integral in (20), it is helpful to separate from it the Fock contribution, which is proportional to χ_0 . Then after simple manipulations, (20) is transformed to

$$R^0(\mathbf{k}) = \varphi(\mathbf{k}) - (1/2) \frac{d^2}{d\rho^2} \times \int \mathcal{V}(\mathbf{q} - \mathbf{k}) \frac{\chi_0^2(q)R(q)}{(1 - R(q)\chi_0(q))} \frac{d^4q dg}{(2\pi)^4 g}, \quad (21)$$

where

$$\varphi(\mathbf{k}) = \mathcal{V}(\mathbf{k}) - \frac{1}{2} \frac{d^2}{d\rho^2} \int \mathcal{V}(\mathbf{q} - \mathbf{k}) \chi_0(q) \frac{d^4q}{(2\pi)^4}. \quad (22)$$

The integration over q_0 can be performed without difficulty, and we obtain the result¹⁴

$$\varphi(\mathbf{k}) = \mathcal{V}(\mathbf{k}) + \frac{1}{2} \frac{d^2}{d\rho^2} \rho \int_0^{2p_F} \int_{-1}^1 \mathcal{V}(\mathbf{q} - \mathbf{k}) \times \left(\frac{3q}{4p_F} - \frac{q^3}{16p_F^2} - 1 \right) \frac{q^2 dq dx}{(2\pi)^2}. \quad (23)$$

As a rule, the remaining integrations can also be performed analytically. Thus, for a δ -function potential $\mathcal{V}_0(k) \equiv \mathcal{V}_0$ we obtain the standard Hartree-Fock result $\varphi(k) = \mathcal{V}_0/2$.

The formula of the local approximation for the compressibility differs from the general formula (13) only in having R^0 , the effective potential of the local approximation, in the integrand instead of the exact effective potential R :

$$K = K^0 - \frac{1}{2} \frac{d^2}{d\rho^2} \int \mathcal{V}(q) \times \frac{\chi_0^2(q)(R(q) - R^0(q))}{1 - R^0(q)\chi_0(q)} \frac{d^4q}{(2\pi)^4} \frac{dg}{g}. \quad (24)$$

Here, K^0 is the compressibility calculated in the local approximation:

$$K^0 = \frac{2}{3} \varepsilon_F^0 - \frac{1}{2} \frac{d^2}{d\rho^2} \int \int_c \mathcal{V}(q) \times \frac{\chi_0(q)}{(1 - R^0(q)\chi_0(q))} \frac{dg}{g} \frac{d^4q}{(2\pi)^4}. \quad (25)$$

The integrand here usually decreases quite rapidly with increasing q and q_0 , and at small q and q_0 the difference $R - R^0$ is small. It is this circumstance that is one of the main reasons for the success of the local approximation in describing the state of physical systems (see below). The problem of calculating the compressibility can be looked at from a very slightly different point of view. It is easy to show that $R^0(0) = d^2(E_0^0 - \tau)/d\rho^2$, where E_0^0 is the ground-state energy, also calculated in the framework of the local approximation. On the other hand, the general analytic properties of the response function χ are in no way violated by the local approximation; in particular, the dispersion relation between the real and imaginary parts of χ , given in Appendix A, still holds. Therefore, the compressibility sum rule (A5) also holds. Concluding this discussion, we can say that in place of the exact functional equation for χ we have obtained a relatively simple integro-differential equation without violating any of the basic consistency conditions of the microscopic theory. We note that in several approaches, for example, in the cluster approach, there is no automatic fulfillment of the compressibility sum rule.

Since the local approximation (17) becomes exact in the limit $k \rightarrow 0$, the accuracy of calculations in its framework will be higher, the more rapidly the potential $\mathcal{V}(\mathbf{k})$ decreases. On the basis of this fact, we must expect that a Coulomb plasma, for which $\mathcal{V}(\mathbf{k}) = 1/k^2$, will be described by this approximation better than a gas of hard balls, for which $\mathcal{V}(\mathbf{k}) = \text{const}$. As we shall see later, this conclusion is confirmed by numerical calculations.

The local approximation also has its shortcomings, but before we discuss them, let us see how the approximation works for definite examples of systems for which Monte Carlo calculations of the equation of state are available.

3. COMPARISON OF RESULTS OF NUMERICAL CALCULATIONS OF THE EQUATION OF STATE OF A SYSTEM IN THE LOCAL APPROXIMATION WITH MONTE CARLO CALCULATIONS

a) Neutron system with repulsive Bethe interaction

In this section, we shall demonstrate in some examples the effectiveness of the local approximation and compare the equations of state $\varepsilon^0(\rho)$ calculated with the effective potential R^0 with the $\varepsilon_{MC}(\rho)$ found by the Monte Carlo methods. Well-known Monte Carlo calculations of the equation of state have been made for the electron gas¹⁸ ("jelly" model) and for a system of neutrons¹⁹ with the Bethe model interaction, which is one of the components (the repulsive core) of the Reid NN potential. For nuclear applications, it is these calculations that have the greatest interest. Therefore, we shall begin with them. The Bethe potential has the form $\mathcal{V}(r) = V_0 e^{-\mu r}/r$ ($V_0 = 9263.1$ MeV·F, $\mu = 4$ F⁻¹).

Calculations in the framework of this approach were made in Ref. 13. The function $\varphi(\mathbf{k})$ can be calculated analytically:

$$\varphi(\mathbf{k}) = -\frac{\pi V_0}{6} \left[-\frac{2}{kp_F} \times \ln \frac{(2p_F - k)^2 + \mu^2}{(2p_F + k)^2 + \mu^2} + \frac{2}{p_F^2} + \frac{(k^2 - 3\mu^2)}{4p_F^4} \times \ln \frac{((2p_F - k)^2 + \mu^2)((2p_F + k)^2 + \mu^2)}{k^2 + \mu^2} + \frac{\mu(\mu^2 - 3p_F^2)}{2kp_F^4} \arctan \frac{2k\mu}{(k^2 + \mu^2)^2/4p_F^2 + \mu^2 - k^2} \right]. \quad (26)$$

To solve Eq. (21), it is necessary to rewrite it as a finite-difference equation:

$$R^0(\mathbf{k}, g_{n+1}) = R^0(\mathbf{k}, g_n) + (g_{n+1} - g_n) \varphi(\mathbf{k}) - \frac{1}{2} \int_{g_n}^{g_{n+1}} \int \mathcal{V}(\mathbf{q} - \mathbf{k}) \times \frac{\chi_0^2(q) R^0(\mathbf{q}, g)}{(1 - R^0(\mathbf{q}, g)\chi_0(q))} \frac{d^4q}{(2\pi)^4} \frac{dg}{g}. \quad (27)$$

Equation (27) for $R^0(g_{n+1})$, for already calculated $R^0(g_n)$, can be solved iteratively, $R^0(g_n)$ being determined right up to the actual value of g . As a result, we obtain the effective interaction R^0 as a function of the variables k , ρ , and g , and, using the expression (1), we find the energy E_0 per unit volume. On the other hand, E_0 can be calculated in terms of the compressibility K on the basis of Eq. (13). Comparison of these two results makes it possible to estimate the accuracy of the numerical solution of Eq. (27). The error in the numerical calculations, determined in this way, is about 3%.

The main results of the calculations are given in Table I. All energies relate to one neutron, and the density of the neutron matter is $\rho = p_F^3/3\pi^2$. The following notation is

TABLE I. Energy of neutron matter calculated in the framework of the local approximation.

p_F, F^{-1}	ρ, F^{-3}	τ, MeV	$w_{\text{HF}}, \text{MeV}$	$\varepsilon_c, \text{MeV}$	$\varepsilon^0, \text{MeV}$	f
1,71	0,17	36,5	231	-181	86,5	4
2,07	0,3	53	424	-305	173	6,3
3,1	1	119	1569	-915	773	18
3,9	2	189	3361	-1632	1919	28
5,1	4	300	7205	-2603	4902	45

also used in the table: τ is the kinetic energy of the noninteracting particles, w_{HF} is the Hartree-Fock potential energy, and $\varepsilon_c^{(0)}$ is the correlation energy. The total energy is $\varepsilon^0 = \tau + w_{\text{HF}} + \varepsilon_c^{(0)}$. It can be seen from the table that at normal densities ($\rho \sim 0.17 F^{-3}$) the correlation part of the energy compensates the Hartree-Fock energy to a considerable degree, so that the kinetic energy is $\tau \sim \varepsilon_c^0$. At high densities, $\varepsilon_c^0 \gg \tau$. Thus, correlations play an important part at all the considered densities, and the gas approximation is invalid. This conclusion is also confirmed by consideration of the compressibility of the system. The contribution of interaction to the compressibility is usually characterized by means of the dimensionless parameter $f = p_F(M/\pi^2)R(0)$. For a gas of hard balls, the constant f must be small: $f \sim p_F a$, where a is the scattering length ($f \rightarrow 0$ as $p_F \rightarrow 0$). But for the considered system, as can be seen from the final column of Table I, which gives the values of f , the relation $f > 1$ holds everywhere. This means that the system cannot be described by the gas approximation. The reason for this behavior of f is simple—the ratio of the Fermi momentum p_F to the parameter μ , which characterizes the range of the forces for the tabulated densities, is not small enough: p_F/μ varies from 1/3 to 1. Table II gives the results of comparison of various approaches to calculation of the energy of the neutron system with the Bethe potential. The first column gives the densities, the second gives the results of the Monte Carlo calculations made in Ref. 19, and the third gives the results of the local approximation¹³ and then the results of the cluster approach²⁰ in the two most successful Pandharipande-Bethe and Jackson-Feenberg versions. The comparison shows that at low densities the cluster

results agree with “experiment”—the Monte Carlo results—somewhat better than the results of the local approximation; at high densities, the results of the local approximation reproduce the experiments better than the Pandharipande-Bethe version but somewhat worse than the Jackson-Feenberg version. It should be borne in mind that the number of cluster calculations is much greater than what is needed in the numerical solution of Eq. (27).

Nuclear liquid consists of protons and neutrons, and therefore it is necessary to consider, in addition to the local interaction, the long-range, Coulomb interaction. A Coulomb system—unpolarized and polarized electron gas—has also been considered by Monte Carlo methods. We now compare the results of the local approximation for it with Monte Carlo calculations.

b) Electron gas (“jelly” model)

We begin with a homogeneous unpolarized system in a homogeneous positively charged medium, so that the total charge of the complete system is zero. To describe such a system, one usually replaces the density ρ by the parameter $r_s = r_0/a_B$, where $a_B = \hbar^2/Mc^2$ is the Bohr radius. (For real metals, r_s varies from 2 to 6.) The parameter r_s determines the ratio of the electron potential energy, $\sim e^2/r_s$, and kinetic energy, $\sim \hbar^2/2Mr_s^2$, and in the limit $r_s \rightarrow 0$ the properties of the system can be calculated in the framework of perturbation theory. For large $r_s > 1$, standard perturbation theory is no longer valid, the random-phase approximation also begins to falsify quite rapidly, but the local approximation still works quite well, and, as calculations show, Eq. (21) can be solved by the iterative procedure of

TABLE II. Mean energies of neutron matter calculated by different methods.

ρ, F^{-3}	$\varepsilon_{\text{MC}}, \text{MeV}$	$\varepsilon^0, \text{MeV}$	$\varepsilon_{\text{FHNC}}^{(1)}, \text{MeV}$	$\varepsilon_{\text{FHNC}}^{(2)}, \text{MeV}$
0,17	89	85,9	89	90
0,3	175	173	176	177
1	782	773	787	790
2	1976	1919	1847	1994
4	4916	4902	—	—

TABLE III. Correlation energy ε per particle of an unpolarized electron gas.

r_s	$\varepsilon_{\text{RPA}}, \text{ eV}$	$\varepsilon^0, \text{ eV}$	$\varepsilon_{\text{MC}}, \text{ eV}$	$\varepsilon_{\text{CC}}, \text{ eV}$	$\varepsilon_{\text{FHNC}}, \text{ eV}$
1	-2.14	-1.62	-1.62	-1.65	-1.86
2	-1.68	-1.23	-1.23	-1.23	-1.34
3	-1.44	-1.01	-1.01	-1.01	-1.07
4	-1.27	-0.87	-0.87	-0.87	-0.91
5	-1.16	-0.77	-0.77	-0.77	-0.80
6	-1.06	-0.69	-0.69	—	—
10	-0.84	-0.51	-0.51	-0.51	-0.51
20	-0.58	-0.31	-0.31	-0.31	—
50	-0.35	-0.17	-0.16	—	—

Ref. 12. In the first iteration, the exact response function on the right-hand side of (21) is replaced by the function χ_0 given by the relation (8). The result can be calculated analytically:

$$\varphi(\mathbf{k}) = \frac{4\pi e^2}{k^2} - \frac{\pi e^2}{3p_F^2} \left[\frac{k^2}{4p_F^2} \ln \left| 1 - \frac{4p_F^2}{k^2} \right| + \frac{2p_F}{k} \ln \left| \frac{2p_F + k}{2p_F - k} \right| + 1 \right], \quad (28)$$

and Eq. (21) now has the form ($g = e^2$)

$$R^0(\mathbf{k}) = \varphi(\mathbf{k}) - \frac{1}{2} \frac{d^2}{d\rho^2} \times \int \int \frac{4\pi}{|\mathbf{q} - \mathbf{k}|^2} \frac{\chi_0^2(q) R^0(q)}{(1 - R^0(q) \chi_0(q))} \frac{d^4 q dg}{(2\pi)^4}. \quad (29)$$

The first iteration φ_1 for the effective potential R^0 can be calculated by substituting the function $\varphi(\mathbf{k})$ in the right-hand integral part of this equation, the next by substituting φ_1 , etc.; the energy of the system is found by substituting the calculated effective potential R^0 in the basic relation (1). The most interesting quantity is the correlation energy of the system—the difference between the exact energy and the Hartree–Fock energy. In tables, the correlation energy is usually given for one electron. In Table III, we give the results of calculations of this energy made in various studies. By ε_{RPA} we denote the correlation energy calculated in the random-phase approximation, and by ε^0 the solution of Eq. (29). Also given here is the Monte Carlo correlation energy ε_{MC} ,¹⁸ which can be regarded as the experimental value. In the final columns we give the correlation energies ε_{CC} and $\varepsilon_{\text{FHNC}}$, calculated in the framework of the coupled-cluster (exp S) theory and FHNC theory in Refs. 21 and 22. It can be seen from the table that the local approximation gives a good description of the experiments in a wide range of densities, whereas in FHNC theory the region of high densities, where perturbation theory works, is reproduced less well. As regards the comparison with the coupled-cluster (exp S) theory,

the results of the two approaches are about the same, but the amount of mathematical labor in the exp S theory is much greater. Note that, as the calculation of Ref. 12 shows, in the region of actual electron densities—up to $r_s \sim 10$, the first iteration of $\varphi(\mathbf{k})$ already works well.

All the above formulas can be readily generalized to the case when the electron gas is completely polarized (the Monte Carlo results for $\varepsilon_1(\rho)$ are given in the same Ref. 18). All the changes in the expressions reduce simply to the fact that the density ρ_1 is now related to the momentum p_F by the Fermi relation $\rho_1 = p_F^3/6\pi^2$, the response function is $\chi_1^0 = (1/2)\chi_0(p_F)$, where χ_0 is given by the relation (8), and the factor 1/2 arises because there is no summation over the particle spins. The interaction energy W_1 per unit volume of the polarized electron gas is

$$W_1 = W_1^{\text{HF}} - \frac{1}{2} \int \int \mathcal{V}(q) [\chi_1^0/(1 - R_{11}\chi_1^0) + 2\pi\rho\delta(\omega)] \frac{d^4 q dg}{(2\pi)^4 g}, \quad (30)$$

where W_1^{HF} is the Hartree–Fock part of W_1 , and $R_{11} = \delta^2 W / \delta\rho_1 \delta\rho_1$.

On the basis of (30), it is easy to obtain by the standard method the equation of the local approximation for the effective potential:

$$R_{11}^0(k) = \varphi_{11}(k) - \frac{1}{2} \frac{d^2}{d\rho_1^2} \int \mathcal{V}(q - k) \times \left[\frac{\chi_1^0(q) R_{11}(\mathbf{q}) \chi_1^0(q)}{1 - R_{11}(\mathbf{q}) \chi_1^0(q)} - \chi_1^0(q) \right] \frac{d^4 q dg}{(2\pi)^4 g}; \quad (31)$$

$$\varphi_{11}(\mathbf{k}) = \frac{4\pi e^2}{k^2} - \frac{2\pi e^2}{p_F^2} \left(\frac{k^2}{12p_F^2} \ln \left| \frac{4p_F^2 - k^2}{k^2} \right| + \frac{2p_F}{k} \ln \left| \frac{2p_F + k}{2p_F - k} \right| + \frac{1}{3} \right). \quad (32)$$

The results of solution of Eqs. (30) and (31) are given together with the Monte Carlo results of Ref. 18 in Table IV. The notation here is practically the same as in Table III.

TABLE IV. Correlation energy ε per particle of a polarized electron gas.

r_s	ε_{MC} , eV	$\varepsilon^{(0)}$, eV	r_s	ε_{MC} , eV	$\varepsilon^{(0)}$, eV
0,1	-1,70	-1,72	4	-0,41	-0,47
0,5	-1,40	-1,40	5	-0,36	-0,42
1	-0,83	-0,86	6	-0,32	-0,38
2	-0,60	-0,66	10	-0,21	-0,29
3	-0,48	-0,55	20	-0,10	-0,19

In all the above, the Monte Carlo equation of state was used as the experimental equation of state. A natural question arises: Why do we need new methods of calculation if there already exist fairly reliable ones, even if cumbersome? Of course, the Monte Carlo method is very powerful and very necessary, but it is not all-powerful. We know that its shortcomings appear when one must deal with complicated—realistic—interaction potentials of nucleons or atoms. But even in a Coulomb plasma it is not always at its best; for example, Monte Carlo calculations indicate that the first phase transition—a ferromagnetic transition in the electron gas—occurs at $r_s = 70$ (Ref. 18), but even earlier, at $r_s \sim 10$, there arises in this gas a spin density wave with wave vector $k_c \sim p_F$, i.e., an antiferromagnetic phase transition occurs (see below). In the Monte Carlo phase diagram it is absent.

We give a brief summary. For all its simplicity, the local approximation is, as we have seen, quite capable of competing with the remaining microscopic approaches, and this makes it possible to use the method to calculate other properties of Fermi systems—their single-particle spectra and the effective interaction of the particles.

4. CALCULATION OF GROUND-STATE PROPERTIES

We begin with the last. In a nonrelativistic system with spin $S = 0$ the effective interaction is characterized by two scalars: $1_1 \cdot 1_2$ and $\sigma_1 \cdot \sigma_2$. For $S \neq 0$, there is one more: $(\sigma_1 S)(\sigma_2 S)$. Their coefficients, scalar functions of the momentum transfer k , determine the various properties of the Fermi systems, in particular the magnetic and quadrupole moments of odd nuclei.

To calculate the components of the effective particle interaction, it is convenient to introduce the vectors $\rho = (\rho_0, \rho_3)$ and $\delta V_0 = \Psi + \sigma_i \Psi \delta V_i^0$, where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Generalizing the expressions of Sec. 1, we can write ($i = 0, 3$)

$$\delta \rho_i = (\chi_{ik} \delta V_k^0); \quad (33)$$

$$\delta^2 E_0 = - (1/2) (\delta V_i^0 \chi_{ik} \delta V_k^0). \quad (34)$$

For a spinless system, the matrix χ_{ik} is diagonal. Its component χ_{00} is just the ordinary linear response function χ considered in the previous section; χ_{33} is the spin component of the matrix $\hat{\chi}$. It is obvious that in this system the interaction matrix R is also diagonal, and

$$\chi_{00}^{-1} = \chi_0^{-1} - R; \quad \chi_{33}^{-1} = \chi_0^{-1} - R_s. \quad (35)$$

The equation for R was already solved in the previous sections. But there we discussed mainly only the equation of state itself; here we shall first speak about the behavior of the actual function R , since the behaviors of the different components of the effective potential have much in common.

Figure 1 shows two graphs of the function $f(k) = p_F(M/\pi^2)R(k)$ for model neutron systems with different NN -interaction potentials. One is with the Bethe potential (1), $\mathcal{V}(r) = V_0 e^{-\mu r}/r$ ($V_0 = 9263$ MeV·F, $\mu = 4.2$ F $^{-1}$, density $\rho = 0.17$ F $^{-3}$), and the other with the Reid soft-core potential, $\mathcal{V}(r) = [-14.95 \exp(-0.7r) - 2358 \exp(-2.8r) + 9263 \times \exp(-4.9r)]/r$ (density $\rho = 0.02$ F $^{-3}$). It can be seen that in the region of small k the curves behave differently. Whereas in the system with the Bethe potential the effective interaction is repulsive, in the system with the Reid potential it is attractive, and, as the calculations show, with increasing density this attraction increases and $f(0)$ reaches the value -1 , at which the compressibility vanishes, and then exceeds it. At one of the points within the region of integration with respect to q and q_0 the integrand in the equation for R now becomes infinite, and the result of the numerical integration becomes sensitive to the choice of the integration procedure, since one of the reference points may be directly near the pole and introduce too large a contribution, distorting the entire result. We shall

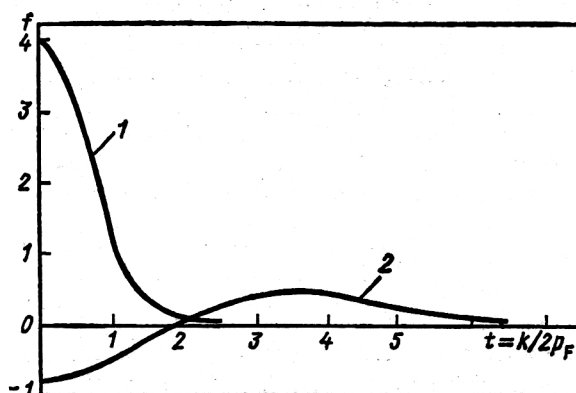


FIG. 1. Dimensionless interaction amplitude in neutron systems.

return to a more detailed discussion of this problem later, and here consider the calculation of the spin-spin component R_s of the effective interaction potential. To find it, we rewrite (34) in the form

$$\delta^2 E_0 = - (1/2) (\delta \rho_i \chi_{ik}^{-1} \delta \rho_k) \quad (36)$$

and we vary (1) twice with respect to ρ_3 . Then we obtain

$$R_s(\mathbf{k}) = - (1/2) \frac{\delta^2}{\delta \rho_3(k) \delta \rho_3(-k)} \times \int \int_C \mathcal{V}(\mathbf{q}) \chi_{00}(q) \frac{d^4 q dg}{(2\pi)^4 g}. \quad (37)$$

The variational derivatives can be calculated by using the matrix formulas $\delta \hat{\chi} = -\hat{\chi} \delta \hat{\chi}^{-1} \hat{\chi}$; $\delta^2 \hat{\chi} = 2\hat{\chi} \delta \hat{\chi}^{-1} \hat{\chi} \delta \hat{\chi}^{-1} \hat{\chi} - \hat{\chi} \delta^2 \hat{\chi}^{-1} \hat{\chi}$. For a spinless system of particles, this gives

$$\frac{\delta^2 \chi_{00}}{\delta \rho_3 \delta \rho_3} = 2\chi_{00} \frac{\delta \chi_{03}^{-1}}{\delta \rho_3} \chi_{33} \frac{\delta \chi_{30}^{-1}}{\delta \rho_3} - \chi_{00} \frac{\delta^2 \chi_{00}^{-1}}{\delta \rho_3 \delta \rho_3} \chi_{00}. \quad (38)$$

We have here used the fact that the function χ_{ik} is diagonal for $S = 0$ (this does not apply to the functional derivatives of $\hat{\chi}!$). We now also use the fact that, in accordance with (36), $\chi_{ik}^{-1} = -\delta^2 E_0 / \delta \rho_i \delta \rho_k$. Varying this equation once more, and then changing the order of variation, we readily obtain

$$\frac{\delta \chi_{03}^{-1}}{\delta \rho_3} = - \frac{\delta^3 E_0}{\delta \rho_3 \delta \rho_3 \delta \rho_0} = \frac{\delta \chi_{33}^{-1}}{\delta \rho_0};$$

$$\frac{\delta^2 \chi_{00}^{-1}}{\delta \rho_3 \delta \rho_3} = - \frac{\delta^4 E_0}{\delta \rho_3 \delta \rho_3 \delta \rho_0 \delta \rho_0} = \frac{\delta^2 \chi_{33}^{-1}}{\delta \rho_0 \delta \rho_0} \quad (39)$$

and

$$\frac{\delta^2 \chi_{33}^{-1}}{\delta \rho_0 \delta \rho_0} = 2\chi_{00} \frac{\delta \chi_{33}^{-1}}{\delta \rho_0} \chi_{33} \frac{\delta \chi_{33}^{-1}}{\delta \rho_0} \chi_{00} - \chi_{00} \frac{\delta^2 \chi_{33}^{-1}}{\delta \rho_0 \delta \rho_0} \chi_{00}.$$

The last formula can be transformed as

$$\frac{\delta^2 \chi_{33}}{\delta \rho_0 \delta \rho_0} = \frac{\chi_{00}^2}{\chi_{33}^2} \frac{\delta^2 \chi_{33}}{\delta \rho_0 \delta \rho_0} = \left(\frac{1 - R_s \chi_0}{1 - R \chi_0} \right)^2 \frac{\delta^2 \chi_{33}}{\delta \rho_0 \delta \rho_0}. \quad (40)$$

Thus, the calculation of R_s can also be reduced to finding the functional derivatives of χ from the ordinary density ρ . We now apply the local approximation to the calculation of the functional derivative $\delta^2 \chi_{33} / \delta \rho_0 \delta \rho_0$. Substituting (40) in (37), we find

$$R_s(\mathbf{k}) = - \frac{1}{2} \int \int \mathcal{V}(\mathbf{q} - \mathbf{k}) \times \left(\frac{1 - R_s(\mathbf{q} - \mathbf{k}) \chi_0(q - k)}{1 - R(\mathbf{q} - \mathbf{k}) \chi_0(q - k)} \right)^2 \times \frac{d^2 \chi_{33}(q)}{d\rho^2} \frac{d^4 q dg}{(2\pi)^4 g}. \quad (41)$$

This equation can be solved with relative ease, by iteration, in an electron gas at real metal densities, i.e., for $r_s \lesssim 6$. The result of an iterative solution is shown in Fig. 2, which gives the r_s dependence of the dimensionless amplitude

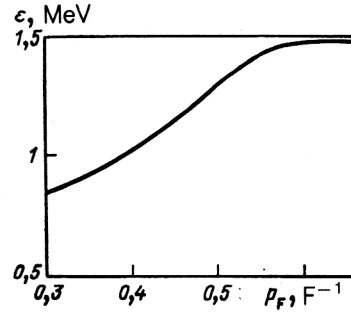


FIG. 2. Binding energy per neutron calculated for the soft-core Reid potential.

$g(0) = (p_F M / \pi^2) R_s(0)$ for $k = 0$. Here too, variation of one of the parameters—in the given case, increase of r_s —makes the minimum of $g(k)$ sink and at a certain point intersect the abscissa -1 . At this point, the spin “rigidity” of the system vanishes. With it, the spin-wave frequency $\omega(k)$ also vanishes and then, in a certain section of wave vectors, $\omega(k)$ is imaginary, as a result of which the system is forced to change its ground state. We defer more detailed discussion of this problem to the section devoted to phase transitions, and here turn to calculation of the effective mass M^* , which characterizes the difference between the spectra of the single-particle excitations of the real Fermi system and of the Fermi gas.

The single-particle state can be regarded as the ground state for nonvanishing momentum \mathbf{p} . If all quantities that occur in the theory are now treated as functionals of the population $n_{\mathbf{p}}$, our theory can be readily generalized and the derivative $\varepsilon(\mathbf{p}) = \delta E_0 / \delta n_{\mathbf{p}}$ can be calculated.

When $p = p_F$, $\varepsilon(\mathbf{p})$ is equal to the chemical potential μ . For $p \neq p_F$, $\varepsilon_{\mathbf{p}} - \mu$ differs from $\varepsilon_{\mathbf{p}}^0 - \mu$ by a factor M/M^* .

To calculate the effective mass, we must take the functional derivative of (1), not with respect to the density ρ , but with respect to the distribution $n_{\mathbf{p}}$. At the same time, $\delta \tau / \delta n_{\mathbf{p}} = \varepsilon_{\mathbf{p}}^0$, and then

$$\varepsilon(\mathbf{p}) = \varepsilon_{\mathbf{p}}^0 - \frac{1}{2} \int \mathcal{V}(\mathbf{q}) \frac{\delta \chi(\mathbf{q})}{\delta n_{\mathbf{p}}} \frac{d^4 q dg}{(2\pi)^4 g}. \quad (42)$$

Using the fact that $\delta \chi = \kappa \delta \chi_0 \kappa + \chi \delta R \chi$, we obtain

$$\varepsilon(\mathbf{p}) = \varepsilon_{\mathbf{p}}^0 - \frac{1}{2} \int \int \mathcal{V}(\mathbf{q}) \left[\kappa(\mathbf{q}) \frac{\delta \chi_0(\mathbf{q})}{\delta n_0} \kappa(\mathbf{q}) + \chi(\mathbf{q}) M(\mathbf{q}, \mathbf{p}) \chi(\mathbf{q}) \right] \frac{d^4 q dg}{(2\pi)^4 g}. \quad (43)$$

An equation for the function $M(\mathbf{q}, \mathbf{p}) = \delta R(\mathbf{q}) / \delta n_{\mathbf{p}}$ is obtained by differentiating with respect to $n_{\mathbf{p}}$ the functional equation (11) for R . In the local approximation, the functional derivatives with respect to ρ are replaced by partial derivatives, but, unfortunately, any replacement of the functional derivatives by partial derivatives (with a corresponding shift of the momenta) leads to the appearance in the equation for $Z(k) = [\delta R(k) / \delta n_{\mathbf{p}}]_{p=p_F}$ of divergences associated with the divergence of the second derivative of $\delta \chi_0 / \delta n_{\mathbf{p}}$ with respect to the density ρ . Let us consider this

problem in more detail, considering first the calculation of the derivative $\delta\chi_0/\delta n_p$, performed in detail in Ref. 23. The first term in $\delta\chi_0/\delta n_p$ arises from standard variation of the product GG with respect to n_p . It takes the form $[\delta\chi_0(k, \omega)/\delta n_p]_I = 2(\epsilon_p^0 - \epsilon_{p-k}^0)/((\epsilon_p^0 - \epsilon_{p-k}^0)^2 - \omega^2)$. After integration over the angles of the vector \mathbf{k} , this gives ($\omega = i\omega$)

$$\left(\frac{\delta\chi_0}{\delta n_p}\right)_I \rightarrow I(k, \omega) = -\frac{M}{k} \frac{d}{dp} \left(\frac{1}{p} \ln \frac{(p+k/2)^2 + \omega^2 M^2/k^2}{(p-k/2)^2 + \omega^2 M^2/k^2} \right). \quad (44)$$

This part of the variational derivative is smooth, and no problems arise with it when the local approximation is applied. However, $\delta\chi_0/\delta n_p$ also contains terms due to the pole avoidance rules on the transition from the ground-state response function χ_0 of the system to the response function χ_0^* of the excited state, with one particle of momentum $p > p_F$ above the Fermi surface or with a hole of momentum $p < p_F$. Direct calculation of this contribution gives²³

$$\left(\frac{\delta\chi_0(q)}{\delta n_p}\right)_{II} = (1 - n_{p-q})\delta(\epsilon_p^0 - \epsilon_{p-q}^0 - \omega)\theta(\omega) - n_{p-q}\delta(\epsilon_p^0 - \epsilon_{p-q}^0 - \omega)\theta(-\omega). \quad (45)$$

After substitution of (44) and (45) in (42), we obtain the equation

$$M/M^* = 1 + \gamma(p) - \frac{1}{2} \int \int \mathcal{V}(q) \{ I(q, \omega) \chi^2(q, \omega) + Z(q, \omega) \chi^2(q, \omega) \} \frac{d^4 q}{(2\pi)^4} \frac{dg}{g}, \quad (46)$$

where

$$\gamma(p) = \frac{M}{4\pi^2 p_F} \int_0^{2p_F} \int_0^g q \mathcal{V}(q) \chi^2(q, \omega=0) dq \frac{dg}{g}. \quad (47)$$

The equation for $Z(k)$ contains the variational derivative $\delta^2\gamma/\delta\rho\delta\rho$. If we attempt to calculate it by means of the local approximation (19), then, as is readily verified, the result of the calculations diverges because of the divergence of the derivative $[d\chi_0/d\rho(q, \omega=0)]_{q \rightarrow 2p_F}$, which occurs in the denominator of $\chi(q, \omega=0)$. The procedure of the calculations can be somewhat modified after the manner in which exact equations for the functional derivatives are derived. We then arrive at the equations

$$Z = Z_1 + Z_2; \quad Z_1(\mathbf{k}) = -\frac{1}{2} \frac{d^2}{d\rho^2} \int \int \mathcal{V}(\mathbf{q}-\mathbf{k}) [I(\mathbf{q}, q_0) \chi^2(\mathbf{q}, q_0) + \chi^2(\mathbf{q}, q_0) Z_1(\mathbf{q})] \frac{d^4 q}{(2\pi)^4} \frac{dg}{g}; \quad (48)$$

$$Z_2(k) = Z_2^{(0)}(k) - \frac{1}{2} \frac{d^2}{d\rho^2} \int \mathcal{V}(\mathbf{q}-\mathbf{k}) \chi^2(\mathbf{q}, q_0) \frac{d^4 q}{(2\pi)^4} \frac{dg}{g}, \quad (49)$$

where

$$Z_2^{(0)}(k) = -\frac{M}{p_F} \frac{d}{dp} \left[\int \int v(q) \left\{ \chi^2(q, \omega) \frac{\delta^2 n_{p-q}}{\delta p(k) \delta p(-k)} + 2 \frac{\delta \chi^2(q, \omega)}{\delta p(-k)} \frac{\delta n_{p-q}}{\delta p(k)} + \frac{\delta^2 \chi^2(q, \omega)}{\delta p(k) \delta p(-k)} n_{p-q} \right\} \frac{d^4 q}{(2\pi)^4} \frac{dg}{g} \right], \quad (49')$$

where the derivative with respect to p is taken at the point $p = p_F$.

Note that for an electron gas at small values of r_s the interaction R can be assumed to be a Coulomb interaction: $R(q) = 4\pi l^2/q^2$. Then the integral with respect to the coupling constant can be calculated, and, ignoring the contribution of Z , we arrive at the usual expression for the effective mass obtained in the random-phase approximation:^{23,24}

$$\frac{M}{M^*} = 1 + \alpha/2 (\ln 1/\alpha - 2); \quad \alpha = e^2/\pi v_F^0.$$

Calculations of M^* in accordance with Eqs. (45)–(49) are currently being made.

5. NUCLEAR MATTER WITH REALISTIC INTERACTIONS

Real nuclear matter differs in the first place from the model systems considered in Sec. 3 by the complicated form of the nucleon–nucleon potential. This very concept can be applied to nuclear matter only at densities at which the nucleon structure is not yet manifested. The characteristic energy for its appearance is, essentially, the distance to the first excited level of the nucleon, i.e., the mass difference between the particle and the nucleon: $\Delta = M_\Delta - M_N \sim 300$ MeV. As long as the Fermi energy is low compared with Δ , the NN interaction can be described by means of the one-boson exchange Hamiltonian. The general form of this Hamiltonian is

$$H_{\text{int}} = \sum g_i \int \bar{\Psi}(\mathbf{r}) O_i \Psi(\mathbf{r}) \varphi_c(\mathbf{r}) d^3 r, \quad (50)$$

where Ψ and $\bar{\Psi}$ are the nucleon annihilation and creation operators, φ is the boson operator, $O = 1$ for a scalar meson, γ_5 for a pseudoscalar meson, etc.

In order to construct the equation of state of the system, i.e., the dependence of the energy ϵ on the density ρ , it is necessary to calculate the expectation value of the Hamiltonian (50) in the ground state. The calculation of this is simplified by the absence of mesons in the ground state. One can rigorously eliminate a contribution of the mesonic degrees of freedom²⁵ by going over from the microscopic Hamiltonian (50) to an effective nucleon–

nucleon Hamiltonian that does not contain bosonic operators, using a procedure like the Hori transformation in QED.²⁶ Hori showed that for calculation of the elements of the S matrix in a system of fermions between states without photons the Hamiltonian $H_{\text{int}} = j_\mu A_\mu$ can be replaced by the four-fermion $H_{\text{eff}} = \frac{1}{2} j_\mu j_\nu D_{\mu\nu}$, where $D_{\mu\nu}(k) = \delta_{\mu\nu}/k^2$ ($k^2 = k_0^2 - \mathbf{k}^2$).

To calculate the energy of the system, Hori's method can be significantly simplified by using the equation of motion for the bosonic operators. We use the well-known Feynman-Hellmann formula

$$g \frac{\partial E_0}{\partial g} = \langle H_{\text{int}} \rangle, \quad (51)$$

where H_{int} is given by the expression (50).

By means of the equations of motion

$$D^{-1}(\hat{k})\varphi(x) = -g_1 \Psi_N^+(x) O_i \Psi_N(x) \quad (52)$$

we eliminate the boson operator φ in the required expectation value:

$$\varphi(x) = -g_i \int D(x-y) \Psi_N^+(y) O_i \Psi_N(y) d^4 y, \quad (53)$$

where $D(k) = (k^2 - \mu^2)^{-1}$ ($D(\mathbf{k}, 0) = -1/(\mathbf{k}^2 + \mu^2)$). After substitution of this expression in (50), we obtain

$$H_{\text{int}} = -g_i^2 \Psi_N^+(x) O_i \Psi_N(x) \Psi_N^+(y) O_i \Psi_N(y) D(x-y). \quad (54)$$

Introducing now the effective coupling constant $\lambda = g_i^2$, we can rewrite (50) in accordance with (51):

$$\lambda \frac{\partial E_0}{\partial \lambda} = \langle H_{\text{eff}} \rangle, \quad (55)$$

where the effective interaction Hamiltonian H_{eff} is determined by

$$H_{\text{eff}} = -\frac{1}{2} \lambda \Psi_N^+(x) O_i \Psi_N(x) \Psi_N^+(y) O_i \Psi_N(y) D(x-y). \quad (56)$$

The resulting expectation value differs from the one in the expression (1) in two respects: 1) the unretarded potential is replaced by the D function, which in principle depends on the frequency ω ; 2) besides the scalar response function λ (in the nonrelativistic limit exchange of σ and ω mesons contributes here), the result also contains a spin response function (due to pion exchange).

It should be said that the existence of contributions to the energy W that depend on spin variables creates great difficulties in any microscopic theory—Monte Carlo results with exchange potentials do not exist, and the cluster expansion has only recently been generalized to the case when the potential \mathcal{V} depends on the total spin of the particles. Let us consider for the example of pion exchange leading to tensor forces how the local approximation comes to terms with this problem. Thus, suppose that σ , ω , and π exchanges are taken into account in the one-boson exchange potential. Then the expression for the total energy of nuclear matter can be written in the form

$$E_0 = E_H - \frac{1}{2} (\mathcal{V} \chi) - \frac{1}{2} (\mathcal{V} \chi_t); \quad (57)$$

$$E_H = \tau + \frac{1}{2} (\rho \mathcal{V} \rho) + \frac{1}{2} (\rho_t \mathcal{V} \rho_t), \quad (58)$$

where χ is the ordinary linear response function, \mathcal{V} is the scalar nonrelativistic potential of σ - ω exchange, $\mathcal{V}_t(k) = f^2[k^2/\mu^2(k^2 + \mu^2)]$ is the one-meson exchange potential ($f^2 = 1.1$), $\chi_t = \delta\rho_t/\delta V_t^0$, $\rho_t = \langle \sigma \mathbf{v} \rangle$, $\delta V_t = \psi^+ \sigma \mathbf{v} \psi \delta V_0$, and $\mathbf{v} = \mathbf{k}/k$. Introducing the two-dimensional vector $\rho = (\rho_0, \rho_t)$, we can write the interaction energy in the form

$$W = - (1/2) (\rho \chi \rho), \quad (59)$$

where the matrix response function χ_{ik} is determined by Eq. (36):

$$\chi_{ik}^{-1} = \delta^2 W / \delta \rho_i \delta \rho_k.$$

We shall consider a neutron system with spin $S = 0$. Then there are no nondiagonal components of the effective interaction, and, as before, we can write $\chi^{-1} = \chi_0^{-1} - R$, while $\chi_t^{-1} = \chi_0^{-1} - R_t$ and

$$R = \frac{\delta^2 n}{\delta \rho \delta \rho}; \quad R_t = \frac{\delta^2 W}{\delta \rho_t \delta \rho_t}. \quad (60)$$

Substituting these relations in (57), we find

$$\left. \begin{aligned} R &= \mathcal{V} - \frac{1}{2} \frac{\delta^2}{\delta \rho \delta \rho} (\mathcal{V} \chi + \mathcal{V} \chi_t); \\ R_t &= \mathcal{V}_t - \frac{1}{2} \frac{\delta^2}{\delta \rho_t \delta \rho_t} (\mathcal{V} \chi + \mathcal{V} \chi_t). \end{aligned} \right\} \quad (61)$$

In the first of these equations, the local approximation can be applied without problems, and we obtain

$$\begin{aligned} R(\mathbf{k}) &= \mathcal{V}(\mathbf{k}) - \frac{1}{2} \frac{d^2}{d\rho^2} \int \int [\mathcal{V}(\mathbf{q} - \mathbf{k}) \chi(q) \\ &\quad + \mathcal{V}_t(\mathbf{q} - \mathbf{k}) \chi_t(q)] \frac{d^4 q dq}{(2\pi)^4 g}. \end{aligned} \quad (62)$$

With regard to the second, two approaches are here possible. The first, and simpler, is related to the fact that the π -meson contribution is small compared with the σ - ω contribution, and therefore the last term in the second equation of (61) can be taken into account in the framework of perturbation theory by replacing χ_t by χ_0 . At the same time, the first term can be transformed in exactly the same way as in the derivation of Eq. (41) for the spin-spin part of the effective particle interaction potential. The upshot is that we obtain an equation of the following form for R_t :

$$\begin{aligned} R_t(\mathbf{k}) &= \mathcal{V}_t(\mathbf{k}) - \frac{1}{2} \frac{d^2}{d\rho^2} \\ &\quad \times \int \left[\left(\frac{1 - R_t(\mathbf{q} - \mathbf{k}) \chi_0(q - k)}{1 - R(\mathbf{q} - \mathbf{k}) \chi_0(q - k)} \right)^2 \frac{d^2 \chi_t}{d\rho^2} \right. \\ &\quad \left. \times \mathcal{V}(\mathbf{q} - \mathbf{k}) + \mathcal{V}_t(\mathbf{q} - \mathbf{k}) \frac{d^2 \chi_0}{d\rho^2}(q) \right] \frac{d^4 q dg}{(2\pi)^4 g}, \end{aligned} \quad (63)$$

which can be solved numerically by the same methods as the equation for R .

A more laborious but simultaneously more accurate way to solve the equations for R_t is to regard ρ_t as a new variable like ρ . Then Eq. (61) for R_t will take the following form in the local approximation:

$$R_t^0 = \mathcal{V}_t - \frac{1}{2} \frac{\partial^2}{\partial \rho_t^2} \int \int [\mathcal{V}_0(\mathbf{q} - \mathbf{k}) \chi_{00}(q) + \mathcal{V}_t(\mathbf{q} - \mathbf{k}) \chi_t(q)] \frac{d^4 q dg}{(2\pi)^4 g}. \quad (64)$$

We obtain as a result two coupled integro-differential equations with partial derivatives. The corresponding numerical calculations are currently being made.

6. STABILITY CONDITIONS AND PHASE TRANSITIONS

The phase diagram of any real system is complicated—it contains a variety of phase transitions: liquid–gas, superconductor–normal metal, etc. Therefore, knowledge of the conditions for stability of a considered state of a system plays an important part in many-body theory. These conditions take the form of the vanishing of the “rigidity” C_η of the system with respect to any type of excitation of it. In the theory of Fermi liquids, there are the well-known stability conditions of Pomeranchuk, which relate vanishing of the rigidity C_η to the corresponding value of the dimensionless local interaction amplitude of quasiparticles at the Fermi surface. Unfortunately, the Landau Fermi-liquid theory, being phenomenological, does not enable one to establish the densities of the system at which particular rigidities C_η vanish.

The situation is different in the present approach. One of its advantages compared with most microscopic theories is that these conditions arise in it naturally, since vanishing of a rigidity means the vanishing of the denominator of the corresponding linear response function. For example, if in a gas the compressibility becomes equal to zero when the density is increased to a certain value ρ_{1c} , then in accordance with (13) this means that at $\rho = \rho_{1c}$ the equation

$$\frac{p_F M}{\pi^2} R(k=\omega=0, \rho_{1c}) = -1 \quad (65)$$

is satisfied for the first time. Equation (65) determines a point of inflection in the curve of $\varepsilon(\rho)$. At large ρ , the energy ε increases quite rapidly, and therefore in the curve of $\varepsilon(\rho)$ there is a further point of inflection ρ_{2c} , at which the compressibility again vanishes. In the range of densities $\rho_{1c} < \rho < \rho_{2c}$ the system is absolutely unstable. As was shown in Ref. 27, neutron matter exhibits such behavior at a low density $\rho \sim 10^{-2} \text{ F}^{-3}$. At such low densities, the contribution of the tensor forces to the binding energy is small because the tensor forces contain an additional square of the momentum transfer, and this is an additional square of the Fermi momentum p_F . Therefore, at low densities it is natural to use the scalar Reid soft-core potential,

$$\mathcal{V}_{\text{RSC}}(r) = [-14.95 \exp(-0.7r) - 2358 \exp(-2.8r) + 9263 \exp(-4.9r)]/r, \quad (66)$$

which ensures good reproduction of the data on low-energy nucleon scattering. (For this potential, the nn scattering length is $a_{nn} = 23 \text{ F}$.)

In calculations made with the potential \mathcal{V}_{RSC} it was found that the curve of the equation of state of rarefied neutron matter contains a point of inflection. It corresponds to a density $\rho_c \sim 0.8 \times 10^{-2} \text{ F}^{-3}$. At higher densities, direct calculations encounter, as we have already said, serious difficulties—on the imaginary axis, on the contour of integration, the response function χ has a pole, in connection with which the result of the calculations of, say, the energy in accordance with Eq. (1) begins to depend on the manner in which the interval of integration is divided up. Nevertheless, textbooks show the curves of the equation of state—for example, van der Waals—as smooth. Therefore, in the microscopic theory it is also desirable to rearrange the calculations in such a way that they give a unique result in the region of absolute instability of the system. Let us consider how this is done in the framework of the given approach. To make the analysis more transparent, we replace the function $\chi_0(z, t)$, which is a combination of logarithms and arctangents, by a simpler asymptotic expression valid when $z, t > 1$:

$$\chi_0^{as}(z, t) = -\frac{p_F M / \pi^2}{1 + 3(z^2 + t^2)}. \quad (67)$$

Then for χ we obtain

$$\chi(z, t) = -\frac{p_F M / \pi^2}{z^2 + \kappa^2(t)}, \quad (68)$$

where $\kappa^2(t) = t^2 + [1 + p_F(M/\pi^2)R(t)]/3$. As long as $\kappa > 0$, the integral over z in (1) gives an arctangent, but if κ becomes negative, this integral, which is always in the principal-value sense, must vanish. On this basis, we can formulate a prescription for taking the integral of χ with respect to z in the expression (1) and others like it, namely, the region of integration in which the denominator of χ is negative makes zero contribution to the result and, therefore, is simply excluded from consideration. In the region in which the denominator of χ is still positive, but small, it is possible to separate the singular part analytically—its contribution is proportional to $\kappa^{-1/2}$, and, although $\kappa \rightarrow 0$ at some value t_0 , the neighborhood of t_c no longer makes a significant contribution to the interval over t . The curve of the equation of state of neutron matter with the nn interaction potential (66) calculated in accordance with this prescription is shown in Fig. 2.

So far, we have been considering a gas–liquid phase transition. However, similar methods can also be used to investigate phase transitions in which the symmetry of the system is spontaneously broken. The example of spin-wave condensation in an electron gas shows how this can be done. The equation for the vanishing of the spin rigidity at some value has a form similar to (65):

$$1 - R_s(k_c, \omega=0, \rho_c) \chi_0(k_c, \omega=0) = 0 \quad (69)$$

with interaction R_s calculated on the basis of (41). In a sufficiently dense gas, it can be solved iteratively. The first iteration φ_s is readily calculated:

$$\varphi_s(k) = -\frac{\pi e^2}{3p_F^2} \left(\frac{k^2}{4p_F^2} \ln \left| \frac{4p_F^2 - k^2}{k^2} \right| + \frac{2p_F}{k} \ln \left| \frac{2p_F + k}{2p_F - k} \right| + 1 \right). \quad (70)$$

To estimate the parameters of the phase transition, we can substitute (70) in (69). Then we find for the critical parameters the values $r_s^c \sim 10$, $k_c \sim p_F$. Allowance for the correction terms shifts these values only slightly.¹²

The antiferromagnetic phase transition in nuclear matter, at which the spin—isospin rigidity $C_{\sigma\tau}$ vanishes at certain values of ρ_c and k_c ,²⁸ can be treated similarly. The change in the sign of $C_{\sigma\tau}$ is caused by tensor forces due to one-pion exchange, and therefore the corresponding transition is usually called the π -condensate transition. In contrast to a transition in an electron system, in which the particles are structureless, in the nuclear antiferromagnetic transition a process with excitation of the Δ resonance, the first nucleon excited state, is predominant. Allowance for such transitions leads, in particular, to the appearance of effective many-particle NN forces,²⁸ as a result of which higher functional derivatives $\delta^2\rho/\delta V_0\delta V_0$, etc., calculated for the Fermi step $n_F = \theta(p - p_F)$, come into play. The final section of the review is devoted to the appropriate generalization of our approach.

In this section we have so far considered phase transitions that are rather similar in kind. They all arise in the neighborhoods of points at which one of the rigidities vanishes, although it must be said that in some cases a more careful analysis reveals that a second-order phase transition may be one of the first order.²⁹

In this series of phase transitions, the transition associated with the change of sign of the effective mass M^* , when the integral term in (42) compensates the free term,³⁰ is out of the ordinary run.

At it, $M/M^* \rightarrow 0$, i.e., initially the fermion “gets heavier,” the spectrum $\varepsilon(p)$ gets flatter, and then even becomes nonmonotonic—a minimum, like the roton in liquid ^3He , appears in it. If in this case the Fermi momentum is increased, then there comes a time when a small increase in the momentum of the quasiparticles at the Fermi surface results in a decrease of the total energy $E_0[n_F(p)]$, and, therefore, the Fermi step $n_F(p) = \theta(p - p_F)$ ceases to describe the ground state of the system—a phase transition occurs. Such a transition may occur in a rarefied electron gas, where, as is well known from the calculations of Ref. 14, the ratio M/M^* decreases with increasing r_s .

Now some words on the microscopic theory of the states that arise after the phase-transition point. To construct such a theory, it is necessary to compare the energies of several competing quasiparticle distributions $n(\mathbf{p})$. To make calculations of the properties of systems after the transition point—and after it some of them become inhomogeneous—we require a certain modification of the approach, which is also needed, in particular, to describe the properties of real nuclei. The following section of the review is devoted to this.

7. REAL NUCLEI

The properties of nuclear systems must be studied in the coordinate representation. Several new problems then arise. First, the self-consistent field $U(\mathbf{r}) = \delta W / \delta \rho(\mathbf{r})$ occurs explicitly in the response function χ_0 , as in the remaining properties of the system. There is a dependence on it, and hence on the coupling constant g in both the density and the kinetic energy τ of the noninteracting particles. To avoid the complications associated with this fact, it was proposed in Ref. 31 to apply to the system an additional external field U_e of amplitude such that the density of the system for the actual value $g = g_0$ of the coupling constant should be the same as for $g = 0$. This can be achieved by taking

$$U_e(\mathbf{r}) = (g_0 - g) \left(\frac{\delta W}{\delta \rho(\mathbf{r})} \right)_{g_0}, \quad (71)$$

i.e., for $g = 0$ it is necessary to apply to the system the external field that acts (as a self-consistent field) for the actual value g_0 , and at $g = g_0$ to apply no field at all. This will mean that both the density $\rho(r)$ and the kinetic energy of the noninteracting particles will be the same for $g = 0$ and $g = g_0$. Since the field U_e is given, the field δH will be proportional to the density of the system, and therefore it will not occur at all in the second variational derivatives of W with respect to ρ , which determine the equation for the effective interaction potential:

$$R(\mathbf{r}, \mathbf{r}') = \frac{\delta^2 W}{\delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}')} \quad (72)$$

This assertion requires some clarification. The response function $\chi(\mathbf{r}, \mathbf{r}', \omega)$ in the finite system is determined by the relation

$$\delta \rho(\mathbf{r}, \omega) = \int \chi(\mathbf{r}, \mathbf{r}', \omega) \delta V^0(\mathbf{r}', \omega) d\mathbf{r}', \quad (73)$$

and the analog of Eq. (4) is

$$\delta^2 E_0 = - (1/2) \int \int \delta V^0(\mathbf{r}, \omega) \chi(\mathbf{r}, \mathbf{r}', \omega) \delta V^0(\mathbf{r}', \omega) d\mathbf{r} d\mathbf{r}'. \quad (74)$$

If we introduce the operator $\chi^{-1}(\mathbf{r}, \mathbf{r}')$, then (74) can be rewritten in a form equivalent to (5), i.e., we find that

$$\chi^{-1}(\mathbf{r}, \mathbf{r}', \omega) = - \frac{\delta^2 E_0}{\delta \rho(\mathbf{r}', \omega) \delta \rho(\mathbf{r}, -\omega)}. \quad (75)$$

Determining the effective potential R by the relation

$$\chi^{-1}(\mathbf{r}, \mathbf{r}', \omega) = \chi_0^{-1}(\mathbf{r}, \mathbf{r}', \omega) - R(\mathbf{r}, \mathbf{r}', \omega) \quad (76)$$

or, equivalently,

$$\begin{aligned} \chi(\mathbf{r}, \mathbf{r}', \omega) &= \chi_0(\mathbf{r}, \mathbf{r}', \omega) \\ &+ \int \chi_0(\mathbf{r}, \mathbf{r}_1, \omega) R(\mathbf{r}_1, \mathbf{r}_2, \omega) \chi(\mathbf{r}, \mathbf{r}', \omega) d\mathbf{r}_1 d\mathbf{r}_2, \end{aligned} \quad (76')$$

by using Eq. (75) we can arrive at the variational formula $R(\mathbf{r}, \mathbf{r}') = \delta^2 W / \delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}')$, and from this we obtain for R the functional equation

$$R(\mathbf{r}, \mathbf{r}', \omega) = \mathcal{V}(\mathbf{r} - \mathbf{r}') - \frac{1}{2} \frac{\delta^2}{\delta \rho(\mathbf{r}, \omega) \delta \rho(\mathbf{r}', \omega)} \times \int \int \mathcal{V}(\mathbf{r}_1, \mathbf{r}_2) \chi(\mathbf{r}_1, \mathbf{r}_2, \omega') d\mathbf{r}_1 d\mathbf{r}_2 d\omega'. \quad (77)$$

Now is the time to discuss the problem of calculating the nuclear density $\rho(\mathbf{r})$. It is measured in experiments in which fast electrons are scattered elastically by nuclei. In self-consistent approaches based on the use of perturbation theory, for example, the Hartree-Fock approach, the particle density $\rho(\mathbf{r})$ is equal to the shell density ρ_{SM} :

$$\rho_{SM}(\mathbf{r}) = \sum n_\lambda \varphi_\lambda^2(\mathbf{r}), \quad (78)$$

where $\varphi_\lambda(\mathbf{r})$ are the eigenfunctions of the shell Hamiltonian $H_{SM} = p^2/2M^* + U_{SM}(\mathbf{r})$, where the effective mass M^* and the self-consistent field $U(\mathbf{r})$ are calculated by means of the Hartree-Fock variational procedure. Analogous expressions for them are also used in the phenomenological Hartree-Fock method with effective forces.³² On the other hand, in the Fermi-liquid approach, even a self-consistent one, expressions of the type (78) for $\rho(\mathbf{r})$ do not exist—it is necessary to introduce an additional unknown local charge $e_q(k)$ of quasiparticles, this being a function of the momentum transfer k that only at $k = 0$ is equal to unity.¹¹ Therefore, in applying the theory of finite Fermi systems to the description of elastic scattering of electrons, one introduces an additional assumption by setting $e_q = 1$ for all momentum transfers k . This shortcoming is not present in the Kohn-Sham method,⁸ in which Eq. (78) is used to determine the particle density, and it can be shown that (78) minimizes the energy of the system if the single-particle Hamiltonian H^{KS} is written in the form

$$H^{KS} = p^2/2M_{KS} + U_{KS}(\mathbf{r}), \quad (79)$$

where $M_{KS} = M$, $U = \delta W / \delta \rho$. The Kohn-Sham single-particle Hamiltonian differs from the Hartree-Fock Hamiltonian in having a kinetic-energy operator of the particles identical to the free operator, since $M_{KS} = M$. Therefore, the spectrum of single-particle levels ε_λ^{KS} has little resemblance to the actual spectrum, but a positive feature is that the Kohn-Sham density $\rho(\mathbf{r})$ can be used to describe electron scattering. At the level of its foundations, the present approach is close to the Kohn-Sham approach, and it is therefore no surprise that in the problem of describing the density $\rho(\mathbf{r})$ the results obtained from the two methods are practically identical even though the chain of arguments in the present approach is not quite the same as in the Kohn-Sham approach.

Let us consider the $\chi(\mathbf{r}_1, \mathbf{r}_2, \omega \rightarrow \infty)$ asymptotic behavior, making the assumption that the limit of $R(\omega \rightarrow \infty)$ is bounded. Since the function $\chi_0(\omega)$ decreases as $\omega \rightarrow \infty$ in proportion to ω^{-2} , the integral term of (76) introduces a

negligibly small contribution, and in the limit $\omega \rightarrow \infty$ we find that χ and χ_0 are equal. On the other hand, the asymptotic behavior of χ can be calculated directly from the Lehmann expansion (A3). Ignoring ω_0 in the denominator of (A3), we can, as is done in a homogeneous system,¹⁴ sum over s by replacing $\omega_0 \rho_{00}$ by the divergence of the current \mathbf{j} , obtaining after calculation of the commutator $[\rho, \text{div } \mathbf{j}]_{00}$ the result

$$\lim_{\omega \rightarrow \infty} \chi(\mathbf{r}_1, \mathbf{r}_2, \omega) = -\Delta \rho(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) / \omega^2. \quad (80)$$

It is obvious that the limit $\chi_0(\omega \rightarrow \infty)$ is the same. But this means that the density of the system of interacting particles is equal to the density of the system of noninteracting particles, i.e., particles moving in the self-consistent field $U(\mathbf{r}) = \delta W / \delta \rho(\mathbf{r})$,

$$\rho(\mathbf{r}) = \sum n_\lambda \psi_\lambda^2(\mathbf{r}), \quad (81)$$

where

$$\left[\frac{p^2}{2M} + U(\mathbf{r}) \right] \psi_\lambda(\mathbf{r}) = \varepsilon_\lambda^0 \psi_\lambda(\mathbf{r}). \quad (81')$$

This is how the problem of calculating the density of nucleons in a nucleus is solved in the present approach. The actual calculations of the functional derivatives $(\delta \rho / \delta \chi_0) \times (\delta^2 \chi_0 / \delta \rho \delta \rho)$ are made in the same way as in a homogeneous system (see Appendix C), though not in the momentum but rather in the coordinate representation.

In actual calculations it is worth beginning, in our view, with the interaction R_∞ calculated for a homogeneous medium. The local approximation will then take the form that the density ρ substituted in this function [it enters it through the Fermi momentum $p_F(\mathbf{r}) = (3\pi^2 \rho(\mathbf{r})^{1/3})$] will be assumed to depend on r in accordance with Eq. (81). Of course, in a more consistent analysis it is necessary to take into account the difference between $R(\mathbf{r}, \mathbf{r}')$ and $R_\infty(\mathbf{r} - \mathbf{r}')$ which arises if for no other reason than the fact that the difference between the free functions χ_0 in finite and infinite systems must be taken into account in Eq. (14). An example of a calculation for a finite system in the framework of the zeroth approximation is given in Ref. 33, in which the Coulomb energy of nuclei is calculated. The calculation makes it possible (qualitatively as yet) to explain the origin of the well-known Nolen-Schiffer anomaly in the Coulomb energies of mirror nuclei as due to the influence of correlation effects.

In the framework of our approach, it is also possible to calculate the changes in the nuclear neutron and proton densities resulting from the addition of one or several nucleons. The expression obtained from (81) and (81') for the change in the density $\delta \rho = \delta_0 \rho + \chi_0(\delta U / \delta \rho) \delta \rho$ or

$$\delta \rho(\mathbf{r}) = \delta_0 \rho(\mathbf{r}) + \int \chi_0(\mathbf{r}, \mathbf{r}_1) R(\mathbf{r}_1, \mathbf{r}_2) \delta \rho(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (82)$$

is essentially the same as in the theory of finite Fermi systems¹¹ or in the Hartree-Fock method with effective

forces, except that for the quasiparticles used here the effective mass is equal to the base mass, and the entire contribution of the residual interaction, as in the Kohn-Sham theory of the electron gas,⁸ is included in the effective particle interaction potential. An expression analogous to (82) can also be written down for the change in the spin density $\rho_s(\mathbf{r})$ of a nucleus on the addition of an odd nucleon. The only difference is that the equation contains the spin component of the effective potential in place of the scalar component. Knowledge of $\delta\rho_s$ makes it possible to calculate the magnetic dipole and octupole moments of nuclei.

8. DISCUSSION OF THE RESULTS. EXTENSION OF THE STANDARD LOCAL APPROXIMATION

As we have seen, the local approximation works rather well under surprisingly diverse conditions. Some of the actual reasons for this were analyzed above, but there are more general reasons too. If one examines the algebraic structure of the system found in Appendix B, then it is readily seen that the algebraically symbolic equations for the operators O^n with $n \geq 3$ are linear, the equation for O^2 is quadratic, and the equation for O^1 is cubic. Suppose that we wish to solve the algebraic equations—quadratic for O^2 or cubic for O^1 —by iteration, representing the solutions in the form of continued fractions. As clarification, let us consider, for example, the quadratic equation $x^2 - ax + b = 0$ with discriminant $D = a^2 - 4b$ and write it in the form

$$x = b/(a - x).$$

The iterative sequence of solutions of this equation—the continued fraction $x_0 = b/a$, $x_1 = b/(a - b/a)$, $x_2 = b/[a - b/(a - b/a)]$ —converges very rapidly in a wide range of variation of the parameters a and b . A solution of the cubic equation can be constructed similarly. We now turn to the symbolic equations for O^n . Solving them, we can represent each of these operators as a sum of continued fractions. After substitution of these sums in the equation for the effective potential R , it also becomes a sum of certain continued fractions. From this point of view, Eq. (20) of the local approximation is then obtained when the operators O^n are replaced everywhere in these sums by their approximate expressions in accordance with the prescription (17): $O^1 \rightarrow O^2(dR/d\rho)(d^2R/d\rho^2) \rightarrow \dots$, etc., and, in addition, all the arguments of the functions in the continued fractions are equated to a single value: $q + k$ [which also makes it possible to obtain the expression $d^2\chi(q + k)/d\rho^2$ in Eq. (20)]. Each of the terms of the sum of continued fractions that determines the operators O^n converges quite rapidly, like the entire algebraic continued fraction; moreover, none of the terms is ever discarded, it is merely that the operators O^n with sufficiently large numbers are replaced in the terms by their approximate expressions in accordance with (17). However, if one or several terms of this sum is ignored, the accuracy of the calculation is significantly reduced. This is why it is bad to truncate the system of equations for the operators O^n —we lose the contribution of the terms that contain

O^{n+1} and O^{n+2} . The local approximation takes into account this contribution in some form, and this gives it definite advantages over the standard prescription for solving an infinite hierarchy of coupled equations by truncating the hierarchy at a particular step.

As we have already said, any iteration of an algebraic equation represented by a continued fraction approximates its true solution with roughly the same degree of accuracy in a wide range of variation of the parameters of the equation. Basically, the same remark applies to the system of equations for the potential R and its functional derivatives. It is not surprising that its iterative solutions based on the use of the local approximation reproduce, like continued fractions, the “true” (Monte Carlo) results to the same accuracy for a wide range of values of the density ρ and coupling constant g despite the fact that the problem does not contain any perturbation-theory parameter. Of course, if one “looks closer,” one can see that the accuracy of the local approximation is higher, the greater the specific weight of the polarization effects, i.e., the more important the part played by the particle-hole channel. At the same time, in the gas limit, in which polarization effects are unimportant and the integrals are accumulated in the region of A values large compared with the Fermi 4-momenta $k = (\omega, \mathbf{k})$, the accuracy of the local approximation is insufficient.²⁷ We recall in this connection that the local approximation leads automatically to an unretarded effective potential R , which, of course, does not correspond to reality, and it is this discrepancy that is manifested in the gas limit.

A serious mathematical shortcoming of the local approximation is that in it we impose on the functional derivative of the response function certain symmetry properties not inherent in it. Indeed, in accordance with (19) the derivatives $\delta\chi(q)/\delta\rho(k)$ and $\delta\chi(k)/\delta\rho(q)$ are equal. In reality, such equality does not hold at all for the variational derivatives of χ but holds for those of χ^{-1} , since it is χ^{-1} that is the second variational derivative with respect to ρ of the total energy E_0 [see (5)], and therefore

$$\frac{\delta\chi^{-1}(x_1x_2)}{\delta\rho(x)} = \frac{\delta^3E_0}{\delta\rho(x)\delta\rho(x_1)\delta\rho(x_2)} = \frac{\delta\chi^{-1}(x_1x)}{\delta\rho(x_2)}. \quad (83)$$

The second derivative $\delta^2\chi/\delta\rho\delta\rho$ is related to the derivatives of χ^{-1} by

$$\begin{aligned} \frac{\delta^2\chi(x_1x_2)}{\delta\rho(x)\delta\rho(x')} = 2 \left[\int \int \chi(x_1x_3) \frac{\delta\chi^{-1}(x_3x_4)}{\delta\rho(x)} \chi(x_4x_5) \right. \\ \times \frac{\delta\chi^{-1}(x_5x_6)}{\delta\rho(x')} \chi(x_6x_2) - \frac{1}{2} \chi(x_1x_3) \\ \times \frac{\delta^2\chi^{-1}(x_3x_4)}{\delta\rho(x)\delta\rho(x')} \chi(x_4x_2) \Big] \Pi dx_i. \end{aligned} \quad (84)$$

Substituting here the expressions for $\delta\chi^{-1}/\delta\rho$ and $\delta^2\chi^{-1}/\delta\rho\delta\rho$, we obtain, in place of (19),

$$\begin{aligned} \frac{\delta^2\chi(q)}{\delta\rho(k)\delta\rho(-k)} &= 2\chi(q) \frac{\delta\chi^{-1}(q)}{\delta\rho(k)} \chi(q+k) \\ &\times \frac{\delta\chi^{-1}(q+k)}{\delta\rho(-k)} \chi(q) \\ &- \chi(q) \frac{\delta^2\chi^{-1}(q)}{\delta\rho(k)\delta\rho(-k)} \chi(q). \end{aligned} \quad (85)$$

If to calculate the derivatives $\delta\chi^{-1}/\delta\rho$ and $\delta^2\chi^{-1}/\delta\rho\delta\rho$ here we use the local approximation (19) and, having obtained

$$\begin{aligned} \frac{\delta\chi^{-1}(q)}{\delta\rho(k)} &= \frac{\delta\chi^{-1}(q+k)}{d\rho}; \\ \frac{\delta^2\chi^{-1}(q)}{\delta\rho(k)\delta\rho(-k)} &= \frac{d^2\chi^{-1}(q+k)}{d\rho^2}, \end{aligned} \quad (86)$$

substitute this result in (84), and substitute (84) in the basic equation (11) for the effective potential, then the solution of the integro-differential equation constructed in this manner will now depend on the frequency ω . Without violating the symmetry condition (83), we can make this equation more accurate if we calculate the variational derivatives of χ_0 head on (see Appendix C) and use the local approximation only for the derivatives of the potential R . Then the equation for R will take the form

$$\begin{aligned} R(k) &= \mathcal{V}(k) - \frac{1}{2} \int \int \mathcal{V}(q) \left[\chi(q) I^2(q, k) \chi(q) \right. \\ &+ 2\chi(q) I^1(q, k) T(q+k) I^1(q, k) \chi(q) \\ &+ 2\chi(q) I^1(q, k) \chi(q+k) \frac{dR(q)}{d\rho} \chi(q) \\ &+ 2\chi(q) \frac{dR(k+q)}{d\rho} \chi(q+k) I^1(q, k) \chi(q) \\ &+ 2\chi(q) \frac{dR(k+q)}{d\rho} \chi(q+k) \frac{dR(q)}{d\rho} \chi(q) \\ &\left. + \chi(q) \frac{d^2R(q+k)}{d\rho^2} \chi(q) \right] \frac{d^4qdg}{(2\pi)^4g}, \end{aligned} \quad (87)$$

where $T=R\chi$, $I^1(q, k) = \delta\chi_0(q)/\delta\rho(k)$, $I^2(q, k) = \delta^2\chi_0(q)/\delta\rho(k)\delta\rho(-k)$.

The main difference between the basic equation (20) and Eqs. (86) and (87) is that the solution of both the one and the other now depends on the frequency ω . Thus, one of the obvious shortcomings of the local approximation is now eliminated. Further refinements in the procedure for the approximate solution of the functional equation (11) are obtained by considering the three exact equations for the blocks R , O^1 , and O^2 . Naturally, they are not closed, since they also contain the blocks O^3 and O^4 . If the latter are calculated in the framework of the local approxima-

tion, then the considered trio is closed, after which it can be solved by numerical methods.

9. CONTRIBUTION OF THE Δ -RESONANCE DEGREES OF FREEDOM TO THE BINDING ENERGY OF NUCLEAR MATTER

The functional approach developed in the previous sections can be significantly extended by applying it, for example, to the microscopic calculation of superfluid Fermi systems³⁴ and to take into account the contribution of non-nucleonic degrees of freedom, the most important role among which is played by the Δ isobar, to the binding energy and other properties of nuclear matter.²⁵ In this section, we consider the latter problem in detail. As has been shown in numerous investigations, beginning with Ref. 28, the appearance of the virtual Δ isobar in the intermediate states leads to the appearance of effective many-particle nucleon-nucleon forces. A quantitative approach to calculation of the parameters of these forces can be constructed as follows. We write down the nuclear one-boson Hamiltonian with allowance for $N\Delta$ transitions:

$$H_{\text{int}} = \int [g_{NN}^i \rho_{NN}^i(\mathbf{x}) \varphi_i(\mathbf{x}) + g_{N\Delta}^i \rho_{N\Delta}^i(\mathbf{x}) \varphi_i(\mathbf{x})] d^3x, \quad (88)$$

where $\rho_{NN}^i = \Psi_N^\dagger O_i \Psi_N$, $\rho_{N\Delta}^i = \Psi_N^\dagger O_i \Psi_\Delta + \Psi_\Delta^\dagger O_i \Psi_N$.

Here, the terms $\rho_{\Delta\Delta}^i$ are omitted, since, as follows from quark analysis, the constants $g_{\Delta\Delta}^i$ are small compared with g_{NN}^i and $g_{N\Delta}^i$.

For simplicity, we restrict ourselves to a Hamiltonian containing the Δ resonances and scalar bosons (the obvious generalization to the pseudoscalar case will be considered below):

$$H_{\text{int}} = \int [g_{NN} \rho_{NN}(\mathbf{x}) + g_{N\Delta} \rho_{N\Delta}(\mathbf{x})] \varphi(\mathbf{x}) d^3x. \quad (89)$$

To eliminate the bosonic and non-nucleonic degrees of freedom, we use equations of motion written in the symbolic form

$$\begin{aligned} -D^{-1}\varphi &= i[\dot{H}_{\text{int}}, \varphi]; \\ G_\Delta^{-1}\Psi_\Delta &= -[H_{\text{int}}, \Psi_\Delta], \end{aligned} \quad (90)$$

where D and G_Δ are the propagators of the φ bosons and Ψ_Δ fermions, respectively. For the bosons, we have

$$-D^{-1}\psi = g_{NN}\Psi_N^\dagger \Psi_N + g_{N\Delta}(\Psi_N^\dagger \Psi_\Delta + \Psi_\Delta^\dagger \Psi_N) \quad (91)$$

and, therefore,

$$\begin{aligned} H_{\text{int}} &= + \{g_{NN}^2 \Psi_N^\dagger \Psi_N D \Psi_N^\dagger \Psi_N \\ &+ 2g_{NN}g_{N\Delta} \Psi_N^\dagger \Psi_N D (\Psi_N^\dagger \Psi_\Delta + \Psi_\Delta^\dagger \Psi_N) \\ &+ g_{N\Delta}^2 (\Psi_N^\dagger \Psi_\Delta D \Psi_N^\dagger \Psi_\Delta + 2\Psi_N^\dagger \Psi_\Delta D \Psi_\Delta^\dagger \Psi_N \\ &+ \Psi_\Delta^\dagger \Psi_N D \Psi_\Delta^\dagger \Psi_N)\}. \end{aligned} \quad (92)$$

Thus, after the field φ has been eliminated, we obtain a four-fermion Hamiltonian with terms corresponding to emission, absorption, and rescattering of Δ particles (Fig.

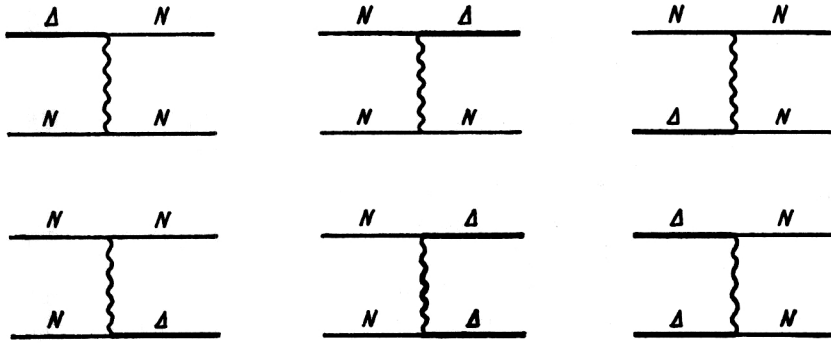


FIG. 3. Simplest diagrams of $N\Delta$ interaction.

3). Note that the Hamiltonian (88) also takes such a form when the meson-isobar interaction is taken into account.

To go over to the nucleon-nucleon Hamiltonian, it "merely" remains to eliminate the fields Ψ_Δ and Ψ_Δ^+ . From Eq. (90), we obtain

$$\Psi_\Delta = g_{N\Delta} G_\Delta \Psi_N \varphi; \quad (93)$$

$$\Psi_\Delta^+ = g_{N\Delta} \tilde{G}_\Delta' \Psi_N^+ \varphi. \quad (93')$$

For convenience of the following calculations, we note that when Eq. (91) is taken into account the Hamiltonian (88) can be written in the form

$$H_{\text{int}} = -\varphi D^{-1} \varphi, \quad (94)$$

and Eq. (91) itself, on substitution in it of (93) and (93'), is closed with respect to φ :

$$\varphi = [g_{NN} D \Psi_N^+ \Psi_N + g_{N\Delta} D (\Psi_N^+ G_\Delta \Psi_N + \Psi_N^+ \tilde{G}_\Delta' \Psi_N) \varphi]. \quad (95)$$

Below, we shall solve Eq. (95) and use it to construct an effective Hamiltonian of the problem.

We introduce the notation

$$\begin{aligned} \lambda_N &= g_{NN}^2; \quad \lambda_\Delta^2 = g_{N\Delta}^2; \\ \hat{\rho} &= \Psi_N^+ \Psi_N; \quad \hat{T} = \Psi_N^+ G_\Delta \Psi_N + \Psi_N^+ \tilde{G}_\Delta' \Psi_N; \\ \eta &= \varphi / g_{NN}. \end{aligned} \quad (96)$$

Then Eq. (95) can be rewritten in the form

$$D^{-1} \eta = \rho + \lambda_\Delta T \eta, \quad (97)$$

and its solution will be

$$\eta = \sum_n \lambda_\Delta^n (D \hat{T})^n D \hat{\rho}. \quad (98)$$

Therefore, the effective nucleon-nucleon Hamiltonian (94) can be written in the form

$$H_{\text{int}} = \lambda_N \sum_{n,k} \lambda_\Delta^{n+k} \hat{\rho} (D \hat{T})^{n+k} D \hat{\rho}. \quad (99)$$

To calculate the contribution of H_{int} to the ground-state energy of the system, we use (94). In the notation of (96), we obtain

$$\delta E_{\text{int}} = \frac{\lambda_N}{2\lambda_\Delta} \int_0^{\lambda_\Delta} \langle \eta D^{-1} \eta \rangle d\lambda'_\Delta. \quad (100)$$

Taking the derivative of both sides of (97), and using the chain of equations

$$\begin{aligned} \eta D^{-1} \eta &= \rho \eta + \lambda_0 \eta T \eta \\ &= \rho \eta + \lambda_\Delta \eta D^{-1} (1 + \lambda_\Delta D T) \frac{\partial \eta}{\partial \lambda_\Delta} \\ &= \rho \eta + \lambda_\Delta \eta (D^{-1} + \lambda_\Delta T) \frac{\partial \eta}{\partial \lambda_\Delta} \\ &= \rho \eta + \lambda_\Delta \rho \frac{\partial \eta}{\partial \lambda_\Delta} \\ &= \rho \frac{\partial}{\partial \lambda_\Delta} (\lambda_\Delta \eta), \end{aligned} \quad (101)$$

we obtain

$$\delta E_{\text{int}} = (1/2) \lambda_N \langle \rho \eta \rangle, \quad (102)$$

where η is determined by the expression (98).

We now recall that the mass difference $M_\Delta - M_N$ is ~ 300 MeV, and $\epsilon_F^0 \sim 40$ MeV. Therefore, in the calculation of the nuclear contributions the dependence of G_Δ on p and ϵ can in a first approximation be ignored, by setting $G_\Delta(p, \epsilon) = G_\Delta = -1/(M_\Delta - M_N)$. Then $\hat{T} = 2G_\Delta \hat{\rho}$ and the contribution to the energy is given by the series

$$\begin{aligned} \delta E_{\text{int}} &= \frac{1}{2} g_{NN}^2 \left\langle \hat{\rho} \sum_n (g_{N\Delta}^2)^n D (2G_\Delta \hat{\rho}) D \dots (2G_\Delta \hat{\rho}) D \hat{\rho} \right\rangle \\ &= \frac{1}{2} g_{NN}^2 \sum_n (2g_{N\Delta}^2 G_\Delta)^n D^n \langle \rho \dots \rho \dots \rho \rangle. \end{aligned} \quad (103)$$

This relation is still valid when the pseudoscalar nature of the pion is taken into account.

Regrouping the terms in the series that gives δE_{int} , we obtain

$$\begin{aligned} \delta E_{\text{int}} &= \frac{1}{2} \frac{g_{NN}^2}{g_{N\Delta}^2} \int \int \sum_{n=2}^{\infty} \chi^{(n)} \\ &\quad \times (q_1 \dots q_n) \prod_i (2G_\Delta g_{N\Delta}^2 D(q_i)) \\ &\quad \times \{1 + 2G_\Delta \rho g_{N\Delta}^2 D(q_1) + 2G_\Delta g_{N\Delta}^2 D(q_i) \\ &\quad \times 2G_\Delta g_{N\Delta}^2 D(q_k) \chi(q, q_n)\} dq_1 \dots dq_n. \end{aligned} \quad (104)$$

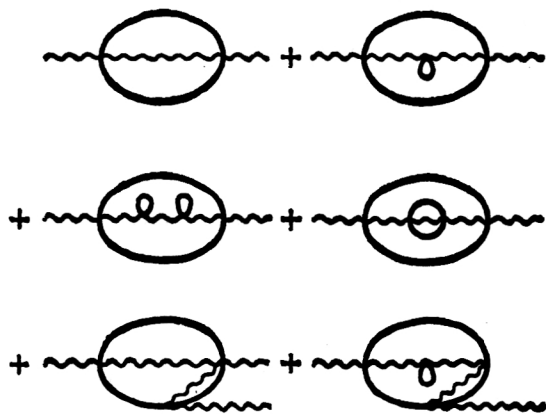


FIG. 4. Simplest contributions to the pion polarization operator.

The series in the curly brackets can be interpreted in terms of Feynman diagrams (Fig. 4).

Collecting together all the reducible diagrams, we obtain the renormalized pion D function in place of the bare function. We emphasize that here the renormalization involves only diagrams of the transition $\pi \rightarrow \Delta + \bar{N}$ and not those of $\pi \rightarrow N + \bar{N}$, which are already included in the corresponding response functions $\chi^{(n)}$. The upshot is that we obtain

$$\delta E_{\text{int}} = \frac{1}{2} \left(\frac{g_{NN}}{g_{N\Delta}} \right)^2 - \frac{1}{2G_{\Delta}} \sum_{n=2}^{\infty} \int dq_1 \dots dq_{n-1} \chi^{(n)} \times (q_1 \dots q_{n-1}) \prod_{i=1}^{n-1} \tilde{D}(q_i), \quad (105)$$

where $\tilde{D} = 2G_{\Delta} g_{N\Delta}^2 D / (1 + 2G_{\Delta} \rho + \dots)$.

We now note that in this section we have nowhere distinguished the effects associated with the presence of the nuclear medium from the vacuum effects. Therefore, (105) contains all the divergences inherent in standard quantum field theory—in the first term of (105), say, we have the integral of the product GG , which contains an ultraviolet divergence that does not depend on the density of the medium and is due to the renormalization of the nucleon mass by the meson field. The prescriptions for eliminating these divergences are the same as in quantum field theory. Calculations of δE_{int} in accordance with Eq. (105) are currently being made.

CONCLUSIONS

In this paper, we have proposed a nonperturbative approach to the calculation of the basic properties of Fermi systems in terms of the potential of the vacuum interaction of the particles. We have written down an exact system of coupled equations for the frequency-dependent effective interaction between quasiparticles in a medium. In contrast to the Jastrow correlation approach, cluster approach, etc., in which one usually introduces a phenomenological two-particle correlation function and determines its parameters by a variational method, such a procedure is completely

absent in the present approach. In this respect too it is a “descendant” of the approaches developed in solid-state theory.

From the system of exact coupled equations, we have obtained approximate but much simpler equations. In the derivation of these equations, which, as we have shown, can be used practically, we have made a number of simplifications, the accuracy of which has largely still to be established. The first problem to be attacked in this direction—comparison of numerical results of this approach based on models with the results of the corresponding Monte Carlo calculations—has in part been completed. The comparison made for two systems—electron liquid and neutron matter—indicates that the method is effective. Unfortunately, it is not yet possible to test in this manner the hierarchy of equations (or at least a pair of equations) because of limitations of the computational technique. However, the system of even two equations would make it possible to obtain much more information—in the first place, the nonlinear response, with which double β decay is associated, microscopic calculations of the parity-nonconserving mean field, etc., and, second, the linear response function that is also obtained in this calculation would make it possible to analyze the resonances of the system, in particular their widths.

If the proposed approach “survives,” then one can consider further investigations at finite temperatures, the generalization of the approach to the description of superfluid systems, and, in particular, for calculation of the gap Δ in terms of vacuum forces. It is possible that the method could be generalized to Bose systems. In nuclear physics there are, besides the two main problems—1) determination of the constants of the theory of finite Fermi systems; 2) microscopic calculation of the equation of state of nuclear matter—some problems for the solution of which this method could be suitable, for example, calculation of the P -odd component of the NN interaction in a nucleus, calculation of the parameters of the self-consistent nuclear potential that acts on the Λ and Σ particles in a nucleus, calculation of the equation of state of neutron matter, and the study of nuclei far from the stability band.

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APPENDIX A

Here, we derive the relation (4), which relates the change δE_0 in the internal energy of the system in a weak external field $\delta V_0(\mathbf{r}, t)$ to the linear response function χ . We shall use the same method as for the derivation of the Lehmann expansion for χ (Ref. 14). We write down the perturbed ground-state wave function Ψ :

$$\Psi(t) = \sum a_n(t) e^{-iE_n t} |n\rangle \quad (A1)$$

with boundary condition $a_n(t \rightarrow \infty) = \delta_{n0}$. Here, $|n\rangle$ is

the exact n th state of the system. In the first order in the external field $\delta V_0 \sim e^{i\omega t - \eta t}$,

$$a_n(t) = \frac{(\rho_k^+)_{n0} \delta V_0(\mathbf{k}, \omega)}{\omega - \omega_{n0} + i\eta} e^{-i(\omega - \omega_{n0})t} - \frac{(\rho_k)_{n0} \delta V_0^*(\mathbf{k}, \omega)}{\omega + \omega_{n0} + i\eta} e^{i(\omega + \omega_{n0})t}, \quad (\text{A2})$$

where $\omega_{n0} = E_n - E_0$, $\rho_k^+ = \sum a_p^+ a_{p+k}$, $\eta \rightarrow 0$.

Substituting (A2) in the expression for ρ and going over to the Fourier transform with respect to the time, we obtain the Lehmann expansion for the response function:

$$\chi(\mathbf{k}, \omega) = \sum |(\rho_k)_{n0}|^2 \left[\frac{1}{\omega - \omega_{n0} + i\eta} - \frac{1}{\omega + \omega_{n0} + i\eta} \right], \quad (\text{A3})$$

in which the imaginary part of χ , which is usually called the dynamical form factor and denoted by S , is represented by the expression

$$S(\mathbf{k}, \omega) = -\text{Im } \chi(\mathbf{k}, \omega) = \sum |\rho_k^+|^2 \delta(\omega - \omega_{s0}). \quad (\text{A4})$$

It follows from this that

$$\chi(\mathbf{k}, \omega) = \int_0^\infty d\omega_1 S(\mathbf{k}, \omega_1) \left[\frac{1}{\omega - \omega_1 + i\eta} - \frac{1}{\omega + \omega_1 + i\eta} \right], \quad (\text{A5})$$

or, otherwise,

$$\text{Re } \chi(\mathbf{k}, \omega) = P \int_0^\infty S(\mathbf{k}, \omega_1) \frac{2\omega_1}{\omega_1^2 - \omega^2} d\omega_1. \quad (\text{A6})$$

Setting here $\omega = k = 0$ and taking into account the relation (12) between $\chi(0)$ and the compressibility of the system, $\chi(0) = -\rho/K$, we find

$$\int S(0, \omega_1) \frac{d\omega_1}{\omega_1} = \frac{\rho}{2K}. \quad (\text{A7})$$

This is the sum rule for the compressibility.¹⁴ Note that the integral over the frequency of $S(k, \omega)$ is called the static form factor $S(k)$. As follows from (A4),

$$S(\mathbf{k}) = (0|\rho_k^+ \rho_k|0) = \frac{1}{\rho} \int_0^\infty d\omega S(\mathbf{k}, \omega). \quad (\text{A8})$$

It can be seen from the definition of the static form factor that it is directly related to the expectation value of the potential-energy operator $H_{\text{int}} = \frac{1}{2} \sum \mathcal{V}(\mathbf{k}) a_{p+k}^+ a_{p_1}^+ a_p a_{p_1+k}$. Substituting here the operators a and a^+ in order to express H_{int} in terms of ρ_k and ρ_k^+ , we find

$$H_{\text{int}} = \frac{1}{2} \sum \mathcal{V}(\mathbf{k}) \rho_k^+ \rho_k - \rho \mathcal{V}(0), \quad (\text{A9})$$

and, thus,

$$\langle H_{\text{int}} \rangle = \frac{1}{2\rho} \int (S(\mathbf{k}) - 1) \mathcal{V}(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3}. \quad (\text{A10})$$

By means of (A8) and (A4), $\langle H_{\text{int}} \rangle$ can be related to the imaginary part of the response function χ , and now, by means of the well-known Feynman-Hellmann formula,

$$g \frac{\partial E}{\partial g} = \langle H_{\text{int}} \rangle, \quad (\text{A11})$$

we can readily obtain the energy ε per particle:

$$\varepsilon = \tau_0 - \frac{1}{2\rho} \int \int \mathcal{V}(\mathbf{k}) [\text{Im } \chi(\mathbf{k}, \omega) + 2\pi\rho\delta(\omega)] \frac{d^4(k)}{(2\pi)^4}, \quad (\text{A12})$$

where $\tau_0 = 3\varepsilon_F^0/5$ is the kinetic energy of the system per particle. The integral along the real axis of ω can be replaced by the integral along the imaginary axis, where the function χ is real (for details, see Ref. 14), and as a result the expression (1) is obtained.

We now calculate the expectation value of the Hamiltonian of the system, $(\Psi_0(t)|H|\Psi_0(t))$, taking into account as well the variation of the coefficient $a_0(t)$ in the second order of perturbation theory. Then

$$\begin{aligned} \delta^2 E_0 &= (\Psi_0|H|\Psi_0) - E_0 \\ &= - \sum \frac{|(\rho_k^+)_{0s}|^2 \delta V_0(\mathbf{k}, \omega) \delta V_0^*(\mathbf{k}, \omega)}{(\omega + i\eta)^2 - \omega_{s0}^2}. \end{aligned} \quad (\text{A13})$$

Comparing (A3) with (A13), we finally obtain

$$\delta^2 E_0 = -\frac{1}{2} \delta V_0(\mathbf{k}, \omega) \chi(\mathbf{k}, \omega) \delta V_0^*(\mathbf{k}, \omega). \quad (\text{A14})$$

APPENDIX B

In this Appendix, we obtain a system of coupled equations for the effective potential $R(k, \omega)$ and its functional derivatives. We begin with the exact equation (10):

$$\begin{aligned} R(k) &= \mathcal{V}(k) \\ &- \frac{1}{2} \frac{\delta^2}{\delta\rho(k)\delta\rho(k)} \int \mathcal{V}(\mathbf{q}) \chi(\mathbf{q}) \frac{d\mathbf{g}}{g} \frac{d^4 q}{(2\pi)^4}. \end{aligned} \quad (\text{B1})$$

To calculate the functional derivatives of χ , we write down the system of variational relations derived directly from the definition of the functions χ and κ [see Eqs. (7) and (8)]. We have

$$\begin{aligned} \delta\chi &= \kappa \delta\chi_0 \kappa + \chi \delta R \chi; \\ \delta\kappa &= \kappa \delta R \chi + T \delta\chi_0 \chi; \\ \delta T &= T \delta\chi_0 T + \kappa \delta R \kappa, \end{aligned} \quad (\text{B2})$$

where we have introduced the scattering amplitude T , which satisfies the equation

$$T = R + R \chi_0 T, \quad (\text{B3})$$

which has in compact form the solution $T = R\chi = R/(1 - R\chi_0)$.

On the basis of (B2), we can also readily calculate the second variation of χ :

$$\delta^2\chi = \kappa\delta^2\chi_0\kappa + 2\kappa\delta\chi_0 T\delta\chi_0\kappa + 2\kappa\delta\chi_0\delta R\chi + 2\chi\delta R\kappa\delta X_0\kappa + 2\chi\delta R\chi\delta R\chi + \chi\delta^2 R\chi. \quad (B4)$$

After substitution of these relations in (B1), we obtain the equation for the effective potential R given in the main text of the review:

$$R(k) = \mathcal{V}(k) - \frac{1}{2} \int \int \mathcal{V}(q) [\kappa(q) I^2(q, k) \kappa(q) + 2\kappa(q) I^1(q, k) T(q + k) I'(q, k) \kappa(q) + 2\kappa(q) I^1(q, k) \kappa(q + k) O^1(q, k) \chi(q) + 2\chi(q) O^1(q, k) \kappa(q + k) I^1(q, k) \kappa(q) + 2\chi(q) O^1(q, k) \chi(q + k) O^1(q, k) \chi(q) + \chi(q) O^2(q, k) \chi(q)] \frac{d^4 q dg}{(2\pi)^4 g}. \quad (B5)$$

Here, we have introduced the notation

$$I^1(q, k) = \frac{\delta\chi(q)}{\delta\rho(k)}; \\ I^2(q, k) = \frac{\delta^2\chi(q)}{\delta\rho(k)\delta\rho(-k)}; \\ O^1(q, k) = \frac{\delta R(q)}{\delta\rho(k)}; \\ O^2(q, k) = \frac{\delta^2 R}{\delta\rho(k)\delta\rho(-k)}. \quad (B6)$$

The factor 2 in front of all the terms in (B5) except the first and the last arises as a result of the sum over the transposition of the momenta k and $-k$.

Thus, the effective potential R of the two-particle interaction can be expressed in terms of the blocks of the effective three- and four-particle interactions O^1 and O^2 . One can also derive equations for these blocks by varying (B1) two more times, but then new variational derivatives O^3 , O^4 , etc., appear. Ultimately, we obtain an infinite sequence of integral equations for the effective amplitudes of the many-quasiparticle interactions. Any of the equations of this system contains just one integration over the intermediate 4-momentum q , and in place of the cumbersome exact Green's functions with complicated analytic properties they contain relatively simple propagators of free particles moving in the self-consistent field $U = \delta W / \delta\rho$.

With increasing number of variational differentiations, the number of terms in the equations rapidly increases, and, to make the resulting expressions less cumbersome, it is helpful to introduce symbolic notation. We denote by I^i the i th variational derivative of χ_0 , and by O^i the corresponding variational derivative of R . It is easy to see that on the right-hand side of the equation for the block R there arises a sum of different products containing the blocks I^i and O^n ($l + n \leq i + 2$) and that: 1) the amplitude T always appears between any two blocks I^i and I^j ; 2) between two O^i of any blocks I^i and O^j there is always the effective

field κ , and between two O^j the response function χ ; 3) if a term begins with the "letter" κ , then it is followed by a block I , and if it begins with the "letter" χ , then it is followed by a block O . It ends with the "letter" κ if in front of it there is a block I , or with the letter χ in the case of a block O (this is a trivial consequence of the requirement of Hermiticity of the result); 4) the sum is taken over all nonidentical transpositions of the blocks O and I of the superscripts i and j and incoming momenta k_i ; 5) the argument of the potential \mathcal{V} in the integrand is always equal to the current 4-momentum q ; in the functions T , κ , and χ in the integrand it is shifted. This shift is determined by the 4-momentum absorbed up to this time, i.e., by the sums of all the arguments in the densities $\rho(k_i)$ with respect to which variation is performed in the O and I blocks in front of the considered block [the sum of the arguments of all the densities $\rho(k_i)$ in the variational derivative $\delta^n W / \delta\rho(k_1)\delta\rho(k_2)\dots\delta\rho(k_n)$ must be zero by virtue of the homogeneity of the system in space and time, i.e., $\Sigma_i k_i = 0$]; 6) an integration is performed over the single current momentum q and over the coupling constant g . Following these rules, we can readily express in symbolic form the equation for the variational derivative:

$$O^1(k_1, k_2) = \frac{\delta R(k_1)}{\delta\rho(k_2)} = \frac{\delta^3 W}{\delta\rho(k_1)\delta\rho(k_2)\delta\rho(-k_1 - k_2)}.$$

We shall use brackets to denote a summation and an integration over all intermediate variables and permutations of the momenta. Then

$$O^1 = -\frac{1}{2} (\mathcal{V} [\kappa I^1 T I^1 T \kappa + \kappa I^1 T I^1 \kappa O^1 \chi + \kappa I^1 \kappa O^1 \kappa I^1 \kappa + \chi O^1 \kappa I^1 \kappa O^1 \chi + \chi O^1 \chi O^1 \kappa I^1 \kappa + \chi O^1 \chi O^1 O^1 \chi + \kappa I^1 T I^2 \kappa + \kappa I^2 T I^1 \kappa + \kappa I^1 \kappa O^2 \chi + \chi O^2 \kappa I^1 \kappa + \chi O^1 \kappa I^2 \kappa + \kappa I^2 \kappa O^1 \chi + \chi O^1 \chi O^2 \chi + \chi O^2 \chi O^1 \chi + \kappa I^3 \kappa + \chi O^3 \chi]). \quad (B7)$$

The expression of the following equations for O^2 , O^3 , etc., can be further simplified if one does not write out explicitly the functions χ , κ , T , and \mathcal{V} but assumes that they are always present in the corresponding I and O blocks. Then (B7) can be expressed quite concisely as a sum of zeros and ones:

$$O^1(1^1 1^1 1^1 + 1^1 1^1 0^1 + 1^1 0^1 0^1 + 0^1 0^1 0^1 + 1^2 1^1 + 1^2 0^1 + 0^2 1^1 + 0^2 0^1 + 1^3 + 0^3). \quad (B8)$$

With the block I^n there is associated here the symbol 1^n , and with the block O^n the symbol 0^n . The corresponding expression for O^2 is

$$O^2 \rightarrow (1^1 1^1 1^1 1^1 + 1^1 1^1 1^1 0^1 + 1^1 1^1 0^1 0^1 + 1^1 0^1 0^1 0^1 + 0^1 0^1 0^1 0^1 + 1^2 1^1 1^1 + 1^2 1^1 0^1 + 1^2 0^1 0^1 + 1^2 1^2 + 1^1 0^2 + 0^2 1^1 1^1$$

$$+ 0^2 1^0 1 + 0^2 0^1 0^1 + 0^2 0^2 + 1^3 1 + 1^3 0^1 + 0^3 1^1 + 0^3 0^1 + 1^4 + 0^4 \quad (\text{B9})$$

To return from here to the normal expression, it is necessary, as we have already said, to substitute in (B9) in the necessary order the \mathcal{V} , κ , χ , and T blocks with the arguments that arise as a result of the variational procedure, carry out all the necessary permutations of the "zeros" and "ones," and also the incoming momenta, and, finally, integrate the resulting expression over the coupling constant and the single 4-momentum q . Proceeding in this spirit, one can derive the equation for any block O^n . One of the distinct advantages of the resulting system, which is equivalent to the exact system of equations of the cluster theories, is that we can write in symbolic form any of the equations of the system without writing out the previous equations:

$$O^n \rightarrow \sum (1^{k_1})^{\alpha_1} (1^{k_2})^{\alpha_2} (1^{k_m})^{\alpha_m} (0^{l_1})^{\beta_1} (0^{l_2})^{\beta_2} \dots (0^{l_n})^{\beta_n}. \quad (\text{B10})$$

Here, $k_1 < k_2 < \dots < k_m$ are positive integers, like α, β_k , and $l_1 < l_2 < \dots < l_n$; $(1^{k_1})^{\alpha_1} = 1^{k_1 1^{k_1} \dots 1^{k_1}}$. These numbers are not arbitrary but are related by the one condition $k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_m \alpha_m + l_1 \beta_1 + l_2 \beta_2 + \dots + l_n \beta_n = i + 2$. (B11)

APPENDIX C

To solve the system of equations obtained for the effective potential R and its functional derivatives, we need the blocks $I^n = \delta^n \chi_0 / \delta \rho(1) \dots \delta \rho(n)$. They can be calculated successively—first I^1 , then I^2 , etc:

$$\begin{aligned} I^1(q, k) &= \frac{\delta \chi_0(p)}{\delta \rho(k)} \\ &= \int \left[\frac{\delta G_0(p)}{\delta \rho(k)} G_0(p + q) \right. \\ &\quad \left. + G_0(p) \frac{\delta G_0(p + q)}{\delta \rho(k)} \right] \frac{d^4 p}{(2\pi)^4}. \end{aligned} \quad (\text{C1})$$

As throughout the paper, the integral sign also includes the operation of summation over the spins. The variational derivative $\delta G_0 / \delta \rho$ is made up of two terms. One is related to the circumstance that in the case of variation of ρ with wave vector k the system ceases to be homogeneous, and even the propagator of the noninteracting particles is changed. This contribution can be calculated as follows. When the external field δV_0 is applied, the change in the Green's function of the noninteracting particles is $\delta G_0 = G_0 \delta V_0 G_0$. From this, dividing by $\delta \rho(k)$, we find

$$\frac{\delta_1 G_0(p)}{\delta \rho(k)} = G_0(p) G_0(p + k) \chi^{-1}(k). \quad (\text{C2})$$

The other contribution to the variational derivative of G_0 is due to the change of the self-consistent field U_0 : $\delta U_0 = R \delta \rho$, and

$$\frac{\delta_2 G_0(p)}{\delta \rho(k)} = G_0(p) G_0(p + k) R(k). \quad (\text{C3})$$

Adding (C2) and (C3), and using (6), we obtain

$$\frac{\delta G_0(p)}{\delta \rho(k)} = G_0(p) G_0(p + k) \chi_0^{-1}(k). \quad (\text{C4})$$

Substituting this expression in (C1), we finally find

$$I^1(q, k) = \chi_0^{-1}(k) I_1(q, k), \quad (\text{C5})$$

where $I_1(q, k) = \delta \chi_0(q) / \delta V_0(k)$. In expanded form,

$$\begin{aligned} I_1(q, k) &= \int [G_0(p) G_0(p + k) G_0(p + q + k) \\ &\quad + G_0(p) G_0(p + q) G_0(p + k + q)] \frac{d^4 p}{(2\pi)^4}. \end{aligned} \quad (\text{C6})$$

The block I_1 is now symmetric with respect to permutations of all momenta, since it represents the third variational derivative of the energy with respect to the external fields.

The next variational derivative,

$$I^2(q, k_1, k_2) = \delta I^1(q, k_1) / \delta \rho(k_2), \quad (\text{C7})$$

can be calculated by repeated differentiation of (C1):

$$\begin{aligned} I^2(k_1, q, k_2) &= \chi_0^{-1}(k_1) I_2(k_1, q, k_2) \chi_0^{-1}(k_2) - \chi_0^{-1}(k_1) \\ &\quad \times I_1(q, k_1 + k_2) \chi_0^{-1}(k_1 + k_2) \\ &\quad \times I_1(k_1 + k_2, q) \chi_0^{-1}(k_2), \end{aligned} \quad (\text{C8})$$

where

$$\begin{aligned} I_2(k_1, q, k_2) &= \int [G_0(p) G_0(p + q) G_0(p + q + k_1) \\ &\quad \times G_0(p + q + k_1 + k_2) + G_0(p) G_0(p \\ &\quad + k_1) G_0(p + k_1 + k_2) \\ &\quad \times G_0(p - q) + G_0(p) G_0(p + k_1) \\ &\quad \times G_0(p - q - k_2) G_0(p - q) \\ &\quad + G_0(p) G_0(p + k_2) G_0(p - q - k_1) \\ &\quad \times G_0(p - q) + G_0(p) G_0(p - q - k_1 \\ &\quad - k_2) \\ &\quad \times G_0(p - q - k_1) G_0(p - q) \\ &\quad + G_0(p) G_0(p - q - k_1 - k_2) G_0(p \\ &\quad - q - k_2) \\ &\quad \times G_0(p - q)] \frac{d^4 p}{(2\pi)^4}. \end{aligned} \quad (\text{C9})$$

As an illustration, we calculate by different methods the effective Hartree-Fock potential $R_{\text{HF}}(k)$. By definition, in the Hartree-Fock approximation

$$\chi_{\text{HF}}(k) = \chi_0(k) + \chi_0(k) R_{\text{HF}}(k) \chi_0(k). \quad (\text{C10})$$

On the other hand, the Hartree-Fock response function $\chi_{\text{HF}}(k)$ can be calculated head on by using the formulas of quantum-mechanical perturbation theory:

$$\begin{aligned} \chi_{\text{HF}}(k) = & \int G_0(p) \mathcal{T}_{\text{HF}}(k) G_0(p+k) \frac{d^4 p}{(2\pi)^4 i} \\ & + \int [(G_{\text{HF}}(p) - G_0(p)) G_0(p+k) \\ & + G_0(p) G_{\text{HF}}(p+k) - G_0(p)] \frac{d^4 p}{(2\pi)^4 i}. \quad (\text{C11}) \end{aligned}$$

The Hartree-Fock vertex part \mathcal{T}_{HF} is determined by the integral

$$\begin{aligned} \mathcal{T}_{\text{HF}}(k) = & 1 + \int G(p) \mathcal{V}(k) G(p+k) \frac{d^4 p}{(2\pi)^4 i} \\ & - \int \int G(p) \mathcal{V}(p-p_1) G(p_1+k) \\ & \times \frac{d^4 p_1 d^4 p_2}{(2\pi)^8}, \quad (\text{C12}) \end{aligned}$$

while the Hartree-Fock Green's function is

$$G_{\text{HF}}(p) = [\varepsilon - \varepsilon_p^0 - \Sigma_{\text{HF}}(\mathbf{p})]^{-1},$$

where

$$\sum_{\text{HF}}(\mathbf{p}) = - \int \mathcal{V}(\mathbf{p} - \mathbf{p}_1) G_0(p_1) \frac{d^4 p_1}{(2\pi)^4}. \quad (\text{C13})$$

Substituting these results in (C11) and comparing with (C10), we find

$$\begin{aligned} R_{\text{HF}}(k) = & \mathcal{V}(k) - \int \chi_0(\mathbf{p}, k) \\ & \times \mathcal{V}(\mathbf{p} - \mathbf{p}_1) \chi_0(\mathbf{p}_1, k) \frac{d^3 p_1 d^3 p_2}{(2\pi)^6} \\ & - \int G_0(p) G_0(p) G_0(p+k) \\ & \times \mathcal{V}(\mathbf{p} - \mathbf{p}_1) G_0(p_1) \frac{d^4 p d^4 p_1}{(2\pi)^8}, \quad (\text{C14}) \end{aligned}$$

where $\chi_0(\mathbf{p}, k) = (n_{\mathbf{p}+\mathbf{k}} - n_{\mathbf{p}}) / (\varepsilon_{\mathbf{p}+\mathbf{k}}^0 - \varepsilon_{\mathbf{p}}^0 - \omega)$.

The same formula can also be obtained differently by varying twice the Hartree-Fock energy E_{HF} with respect to ρ and using Eq. (C5) for the derivative $\delta^2 \chi / \delta \rho \delta \rho$:

$$\begin{aligned} R_{\text{HF}}(k) = & \mathcal{V}(k) - \frac{1}{2} \int \mathcal{V}(q) \left[\chi_0^{-2}(k) I_2(q, k) \right. \\ & \left. + \frac{1}{2} \chi_0^{-2}(k) \chi_0^{-1}(0) I_1(0, k) I_1(q, k) \right] \frac{d^4 q}{(2\pi)^4 i}. \quad (\text{C15}) \end{aligned}$$

Substituting here (C6) and (C9), and making simple manipulations, we again arrive at the relation (C14).

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